Optimal and Robust Sliding Mode Regulator for Linear Systems with Delayed Control

Regulador Óptimo y Robusto con Modos Deslizantes para Sistemas Lineales con Control Retardado

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Abstract

This paper presents the optimal regulator for a linear system with time delay in control input and a quadratic criterion. The optimal regulator equations are obtained using the duality principle, which is applied to the optimal filter for linear systems with time delay in observations. Performance of the obtained optimal regulator is verified in the illustrative example against the best linear regulator available for linear systems without delays. Simulation graphs and comparison tables demonstrating better performance of the obtained optimal regulator are included. The paper then presents a robustification algorithm for the obtained optimal regulator based on integral sliding mode compensation of disturbances. The general principles of the integral sliding mode compensator design are modified to yield the basic control algorithm oriented to time-delay systems, which is then applied to robustify the optimal regulator. As a result, the sliding mode compensating control leading to suppression of the disturbances from the initial time moment is designed. The obtained robust control algorithm is verified by simulations in the illustrative example.
Keywords: Linear time-delay system, Optimal control, Filtering, Sliding mode regulator.

Resumen

Este artículo presenta el regulador óptimo para un sistema lineal con retardo en la entrada de control y un criterio cuadrático. Las ecuaciones del regulador óptimo se obtienen usando el principio de dualidad, el cual es aplicado al filtro óptimo para sistemas lineales con tiempo de retardo en las observaciones. El desempeño de el regulador óptimo obtenido es verificado en el ejemplo ilustrativo contra el mejor regulador disponible para sistemas lineales sin retardo. Se incluyen las gráficas de simulación mostrando mejor desempeño del regulador óptimo obtenido. El artículo también presenta un algoritmo de robustificación para el regulador óptimo obtenido basado en compensación de perturbaciones con modos deslizantes integrales. Los principios generales del diseño del compensador con modos deslizantes integrales se modifican para dar el algoritmo básico de control oritnetado a sistemas con retardo en el tiempo. Como resultado, se diseña el control compensador con modos deslizantes llevando a la supresión de las perturbaciones desde el momento del tiempo inicial. El algoritmo de control robusto obtenido es verificado con simulaciones en el ejemplo ilustrativo.
Palabras Clave: Sistemas lineales, Retardo, Control óptimo, Filtrado, Regulador, Modos deslizantes.

1 Introduction

Although the optimal control (regulator) problem for linear system states was solved, as well as the filtering one, in 1960's, [22,15], the optimal control problem for linear systems with delays is still open, depending on the delay type,

specific system equations, criterion, etc. Such complete reference books in the area as [20,21,25,10,7] note, discussing the maximum principle [19] or the dynamic programming method [26] for systems with delays, that finding a particular explicit form of the optimal control function might still remain difficult. A particular form of the criterion must be also taken into account: the studies mostly focused on the time-optimal criterion (see the paper [27] for linear systems) or the quadratic one [13,9,35]. Virtually all studies of the optimal control
in time-delay systems are related to systems with delays in the state (see, for example, [1]), although the case of delays in control input is no less challenging, if the control function should be causal, i.e., does not depend on the future values of the state. A considerable bibliography existing for the robust control problem for time delay systems (such as [12,24]) is not discussed here.

The first part of this paper concentrates on the solution of the optimal control problem for a linear system with delay in control input and a quadratic criterion, which is based on the duality principle in a closed-form situation [3] applied to the optimal filter for linear systems with delay in observations obtained in [5]. Taking into account that the optimal control problem can be solved in the linear case applying the duality principle to the solution of the optimal filtering problem [23], this paper exploits the same idea for designing the optimal control in a linear system with time delay in control input, using the optimal filter for linear systems with delay in observations. In doing so, the optimal regulator gain matrix is constructed as dual transpose to the optimal filter gain one and the optimal regulator gain equation is obtained as dual to the variance equation in the optimal filter. The results obtained by virtue of the duality principle can be rigorously verified through the general equations of the maximum principle [30,20] or the dynamic programming method [7,27] applied to a specific time-delay case, although the physical duality seems obvious: if the optimal filter exists in a closed form, the optimal closed-form regulator should also exist, and vice versa [3].

It should be noted, however, that application of the maximum principle to the present case gives one only a system of state and co-state equations and does not provide the explicit form of the optimal control or co-state vector. So, the duality principle approach actually provides one with the explicit form of the optimal control and co-state vector, which should be then substituted into the equations given by the rigorous optimality tools and thereby verified.

Finally, performance of the obtained optimal control for a linear system with time delay in control input and a quadratic criterion is verified in the illustrative example against the best linear regulator available for linear systems without delays. The simulation results show a definitive advantage of the obtained optimal regulator in both the criterion value and the value of the controlled variable.

The paper then presents a robustification algorithm for the obtained optimal regulator based on integral sliding mode compensation of disturbances. Conventional (non-integral) sliding modes are widely used for uncertainties compensation (see, for example, [37]). On the other hand, time delay effects that take place in relay and sliding mode control systems must be taken into account for the systems analysis and control design
[37,2].

It is also known that time delays do not allow to design the sliding mode control in the space of state variables. Moreover, papers [17,18] show that even in the simplest one-dimensional delayed relay control system only oscillatory solutions can occur. That is why there are the two following main research directions in sliding mode uncertainties compensation for delay systems.

a. Time Delay Compensation

Pade approximation of delay reducing the relay delay output tracking problem to the sliding mode control for nonminimum phase systems was suggested in [35]. In [12,24,33], the sliding mode control in the space of predictor variables was realized. However, subsequent research [19,34] has demonstrated that the conventional sliding mode control design in the space of predictor variables

- cannot compensate even for the matching uncertainties;

- in the case of square systems, if the dimensions of state space and control are the same, sliding mode design in the space of predictor variables can remove the uncertainties in the space of predictor variables but cannot guarantee suppression of uncertainties in the space of state variables.

b. Sliding Mode Control Design Via Feedback Control

In a series of papers (see for example, [32,29]), the conventional sliding modes in the space of state variables of delayed systems are specifically used for elimination of uncertainties. An original idea of combining the sliding mode and high-gain observer design for stochastic systems is recently claimed in [31]. Recently, application of the integral sliding mode to time-delay systems has been initiated: in [4], the integral sliding mode is used for robustification of optimal filters over observations with delay.

The second part of this paper presents an integral sliding mode regulator robustifying the optimal regulator for linear systems with multiple time delays in control input. The idea is to add a compensator to the known optimal control to suppress external disturbances deteriorating the optimal system behavior [37,9]. The integral sliding mode compensator is realized as a relay control in a such way that the sliding mode motion starts from the initial moment, thus eliminating the external disturbances from the beginning of system functioning. This constitutes the crucial advantage of the integral sliding modes in comparison to the conventional ones.

Although the optimal control to be robustified is designed for a system with time delay in control input and is causal, i.e., depends on the delayed system state, the sliding mode control-compensator is designed without delay, since there is the only way to make it capable of eliminating external disturbances in real time. This sequence of actions corresponds to the following technical problem. Let us assume that the optimal control program using the delayed control has already been inserted into the actuator (for example, at the factory). However, in field conditions, the system behavior is affected by external disturbances, such as vibrations or industrial pollution. The task of the field team is to eliminate the influence of external disturbances, directing a system trajectory to the optimal one (which is obtained using the delayed optimal control) by all available means. Of course, in this situation, all kinds of control-compensator functions, including those depending on the current time state, could be used. This is exactly a situation where the obtained robust integral sliding mode regulator is helpful.

The paper is organized as follows. Section 2 states the optimal control problem for a linear system with time delay in control input and describes the duality principle for a closed-form situation [3]. For reference purposes, the optimal filtering equations for a linear state and linear observations with delay [5] are briefly reminded in Section 3. The optimal control problem for a linear system with time delay in control input is solved in Section 4, based on application of the duality principle to the optimal filter of the preceding section.
The paper then presents a robustification algorithm for the obtained optimal regulator based on integral sliding mode compensation of disturbances [37]. Section 5 outlines the
general principles of the integral sliding mode compensator design, which yield the basic algorithm applied then to robustify the optimal regulator. As a result, the sliding mode compensating control leading to suppression of the disturbances from the initial moment is designed in Section 6. This control algorithm actually guarantees all-time coincidence of the disturbed state with the optimally controlled one. Section 7 presents an example illustrating the quality of control provided by the obtained optimal regulator for linear systems with time delay in control input against the best linear regulator available for systems without delays. Simulation graphs and comparison tables demonstrating better performance of the obtained optimal regulator are included. This section then presents an example illustrating the quality of disturbance suppression provided by the obtained robust integral sliding mode regulator against the optimal regulator under the presence of disturbances. Satisfactory results are obtained.

2 Optimal Control Problem for Linear System with Time Delay in Control Input

Consider a linear system with time delay in control input

$$\dot{x}(t) = (a_0(t) + a(t)x(t))dt + B(t)u(t-h)dt, \tag{1}$$

with the initial condition $x(s) = (s)$, $s \in [-h,0]$, where $x(t) \in \Re^n$ is the system state, $u(t) \in \Re^m$ is the control variable, and (s) is a piecewise continuous function given in the interval $[-h,0]$. Existence of the unique solution of the equation (1) is thus assured by the Caratheodori theorem (see, for example, [15]). The quadratic cost function to be minimized is defined as follows:

$$J = \frac{1}{2}\left[x(T) - x_1\right]^T \psi \left[x(T) - x_1\right] +$$
$$\frac{1}{2}\int_{t_9}^T u^T(s)R(s)u(s)ds +$$
$$\frac{1}{2}\int_{t_0}^T x^T(s)L(s)x(s)ds \tag{2}$$

where x_1 is a given vector, R is positive and $,L$ are nonnegative definite symmetric matrices, and $T > t_0$ is a certain time moment.

The optimal control problem is to find the control $u(t)$, $t \in [t_0, T]$, that minimizes the criterion J along with the trajectory $x^*(t)$, $t \in [t_0, T]$, generated upon substituting $u^*(t)$ into the state equation (1). To find the solution to this optimal control problem, the duality principle [23] can be used. For linear systems without delay, if the optimal control exists in the optimal control problem for a linear system with the quadratic cost function J, the optimal filter exists for the dual linear system with Gaussian disturbances and can be found from the optimal control problem solution, using simple algebraic transformations (duality between the gain matrices and between the gain matrix and variance equations), and vice versa (see [23]). Taking into account the physical duality of the filtering and control problems, the last conjecture should be valid for all cases where the optimal control (or, vice versa, the optimal filter) exists in a closed finite-dimensional form [3]. This proposition is now applied to the optimal filtering problem for linear system states over observations with delay, which is dual to the stated optimal control problem (1), (2), and where the optimal filter has already been obtained (see [5]).

3 Optimal Filter for Linear State Equation and Linear Observations With Delay

In this section, the optimal filtering equations for a linear state equation over linear observations with delay (obtained in [5]) are briefly reminded for reference purposes. Let the unobservable random process $x(t)$ be described by an ordinarydifferential equation for the dynamic system state

$$dx(t) = (a_0(t) + a(t)x(t))dt + b(t)dW_1(t), \qquad x(t_0) = x_0, \tag{3}$$

and a delay-differential equation be given for the observation process:

$$dy(t) = (A_0(t) + A(t)x(t-h))dt + F(t)dW_2(t), \tag{4}$$

where $x(t) \in \Re^n$ is the state vector, $y(t) \in \Re^m$ is the observation process, the initial condition $x_0 \in \Re^n$ is a Gaussian vector such that x_0, $W_1(t)$, $W_2(t)$ are independent. The observation process $y(t)$ depends on the delayed

state $x(t-h)$, where h is a fixed delay shift, which assumes that collection of information on the system state for the observation purposes is possible only after a certain time h.

The vector-valued function $a_0(s)$ describes the effect of system inputs (controls and disturbances). It is assumed that $A(t)$ is a nonzero matrix and $F(t)F^T(t)$ is a positive definite matrix. All coefficients in (3)--(4) are deterministic functions of appropriate dimensions.

The estimation problem is to find the estimate of the system state $x(t)$ based on the observation process $Y(t) = \{y(s), 0 \le s \le t\}$, which minimizes the Euclidean 2-norm

$$J = E\left[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))\right]$$

at each time moment t. In other words, our objective is to find the conditional expectation

$$m(t) = \hat{x}(t) = E(x(t) \mid F_t^Y).$$

As usual, the matrix function

$$P(t) = E\left[(x(t) - m(t))^T \mid F_t^Y\right]$$

is the estimate variance.

The solution to the stated problem is given by the following system of filtering equations, which is closed with respect to the introduced variables, $m(t)$ and $P(t)$:

$$dm(t) = (a_0(t) + a(t)m(t))dt + P(t)\exp\left\{(-\int_{t-h}^{t} a^T(s)ds)\right\}A^T(t) \times \tag{5}$$

$$(F(t)F^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t-h))dt).$$

$$dP(t) = (P(t)a^T(t) + a(t)P(t) + b(t)b^T(t) - \tag{6}$$

$$P(t)\exp\left\{(-\int_{t-h}^{t} a^T(s)ds)\right\}A^T(t)(F(t)F^T(t))^{-1} \times$$

$$A(t)\exp\left\{(-\int_{t-h}^{t} a^T(s)ds)\right\}P^T dt.$$

The system of filtering equations (5) and (6) should be complemented with the initial conditions $m(t_0) = E\left[x(t_0) \mid F_{t_0}^Y\right]$ and $P(t_0) = E\left[(x(t_0) - m(t_0)x(t_0) - m(t_0)^T \mid F_{t_0}^Y\right]$. As noted, this system is very similar to the conventional Kalman-Bucy filter, except the adjustments for delays in the estimate and variance equations, calculated due to the

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Cauchy formula for the linear state equation.

In the case of a constant matrix a in the state equation, the optimal filter takes the especially simple form $\exp\left\{(-\int_{t-h}^{t} a^T ds)\right\} dt = \exp\left\{-a^T h\right\}$

$$dm(t) = (a_0(t) + a(t)m(t))dt + P(t)\exp\left\{-a^T\right\}A^T(t) \times \qquad (7)$$

$$(F(t)F^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t-h))dt).,$$

$$dP(t) = (P(t)a^T(t) + a(t)P(t) + b(t)b^T(t) - \qquad (8)$$

$$P(t)\exp\left\{-a^T h\right\}A^T(t)(F(t)F^T(t))^{-1} \times$$
$$A(t)\exp\left\{-a^T h\right\}P^T dt .$$

Thus, the equation (5) (or (7)) for the optimal estimate $m(t)$ and the equation (6) (or (8)) for its covariance matrix $P(t)$ form a closed system of filtering equations in the case of a linear state equation and linear observations with delay.

4 Optimal Control Problem Solution

Let us return to the optimal control problem for the linear state (1) with time delay in linear control input and the cost function (2). This problem is dual to the filtering problem for the linear state (3) and linear observations with delay (4). Since the optimal filter gain matrix in (5) is equal to

$$K_f = P(t)\exp\left\{(-\int_{t-h}^{t} a^T(s)ds)\right\}A^T(t)(F(t)F^T(t))^{-1},$$

the gain matrix in the optimal control problem takes the form of its dual transpose

$$K_c = R(t)^{-1}B^T \exp\left\{(-\int_{t-h}^{t} a^T(s)ds)\right\}Q(t),$$

and the optimal control law is given by

$$u_{opt}(t-h) = K_c x(t-h) = R(t)^{-1}B^T \exp\left\{(-\int_{t-h}^{t} a^T(s)ds)\right\}Q(t)x(t-h), \qquad (9)$$

where the matrix function $Q(t)$ is the solution of the following equation dual to the variance equation (6)

$$dQ(t) = (-a^T(t)Q(t) + -Q(t)a(t) + L(t) -$$

$$Q(t)\exp\left\{(\int_{t-h}^{t} a(s)ds)\right\}B(t)R^{-1}(t) \times$$

$$B^T(t) \exp \left\{ (\int_{t-h}^{t} a^T(s)ds) \right\} Q^T dt \tag{10}$$

with the terminal condition $Q(T) = \quad$.

Upon substituting the optimal control (9) into the state equation (1), the optimally controlled state equation is obtained

$$dx(t) = (a_0(t) + a(t)x(t) + B(t)(R(t)^{-1}B^T(t) \times$$

$$\exp \left\{ (-\int_{t-h}^{t} a^T(s)ds) \right\} Q(t)x(t-h)dt, \quad x(t_0) = x_0.$$

The results obtained in this section by virtue of the duality principle are proved (the proof is given in the Appendix) using the general equations of the Pontryagin maximum principle [29,19]. (Bellman dynamic programming [6,26] could serve as an alternative verifying approach). It should be noted, however, that application of the maximum principle to the present case gives one only a system of state and co-state equations and does not provide the explicit form of the optimal control or co-state vector. So, the duality principle approach actually provides one with the explicit form of the optimal control and co-state vector, which should be then substituted into the equations given by the rigorous optimality tools and thereby verified.

5 Robustification of Motions in the Delay Control Systems Via Integral Sliding Modes

For given control system with delay

$$\dot{x}(t) = f(x(t)) + B(t)u(t-h), \tag{11}$$

where $x \in \Re^n$ is the state vector and $u(t-h) \in \Re^m$ is the control input of rank $B=m$, suppose that there exists a differentiable in x state feedback control law $u_0(x(t-h))$, such that the dynamics of the ideal closed loop system takes the form

$$\dot{x}_0(t) = f(x_0(t)) + B(t)u_0(x_0(t-h)), \tag{12}$$

and has certain desired properties.

However, in practical applications, system (11) operates under uncertainty conditions that may be generated by parameter variations and external disturbances. Let us consider the real trajectory of the closed loop control system

$$\dot{x}(t) = f(x(t)) + B(t)u + g_1(x(t),t) + g_2(x(t-h),t), \tag{13}$$

where g_1, g_2 are smooth uncertainties presenting perturbations and nonlinearities in system (11). For g_1, g_2, the standard matching conditions are assumed to be held: $g_1, g_2 \in \text{span} B$, or, in other words, there exist smooth functions g_1, g_2 such that

$$g_1(x(t), t) = B\gamma_1(x(t), t),$$
$$g_2(x(t-h), t) = B\gamma_2(x(t-h), t),$$
$$\| \gamma_1(x(t), t) \| \le q_1 \| x(t) \| + p_1, \qquad p_1, q_1 > 0,$$
$$\| \gamma_2(x(t-h), t) \| \le q_2 \| x(t-h) \|, \qquad q_2 > 0,$$

The following initial conditions are assumed for system (11)

$$x(\theta) = \varphi(\theta), \tag{14}$$

where $\varphi(\theta)$ is a piecewise continuous function given in the interval [-*h*,0].

Thus, the control problem now consists in robustification of control design in system (12) with respect to uncertainties g_1, g_2: to find such a control law that the trajectories of system (13) with initial conditions (14) coincide with the trajectories $x_0(t)$ with the same initial conditions (14).

5.1 Design Principles

Let us redesign the control law for system (11) in the form

$$u(t-h) = u_0(x(t-h)) + u_1(t), \tag{15}$$

where $u_0(x(t-h))$ is the ideal feedback control designed for (12), and $u_1(t) \in \mathfrak{R}^m$ is the relay control generating the integral sliding mode in some auxiliary space to reject uncertainties g_1, g_2. Substitution of the control law (15) into the system (13) yields

$$\dot{x}(t) = f(x(t)) + B(t)u_0(x(t-h)) + B(t)u_1(t) + g_1(x(t), t) + g_2(x(t-h), t). \tag{16}$$

Define the auxiliary function

$$s(t) = z(t) + s_0(x(t)), \tag{17}$$

where $s_0(x(t)) \equiv u_0(x(t))$, and *z*(*t*) is an auxiliary variable defined below. Then,

$$\dot{s}(t) = \dot{z}(t) + G(t)\big[f(x(t)) + B(t)s_0(x(t-h)) + B(t)(\gamma_1(x(t), t)) + B(t)(\gamma_2(x(t), t)) + B(t)u_1(t) \big], \tag{18}$$

where $G(t) = ds_0(x(t)) / dx(t)$.

The philosophy of integral sliding mode control is the following: in order to achieve $x(t) \equiv x_0(t)$ at all $t \in (-h, \infty)$, the sliding mode should be organized on the surface $s(t)$, since the following disturbance compensation should have been obtained in the sliding mode motion

$$B(t)u_{1eq}(t) = -B(t)(\gamma_1(x(t),t)) - B(t)(\gamma_2(x(t),t)).$$

Define the auxiliary variable $z(t)$ as the solution to the differential equation

$$\dot{z}(t) = -G(t)\left[f(x(t)) + B(t)u_0(x(t-h))\right]$$

with the initial conditions $z(\theta) = -s(\theta) = \varphi(\theta)$, for $\theta \in [-h, 0]$. Then, the sliding manifold equation takes the form

$$\dot{s}(t) = G(t)\left[B(t)(\gamma_1(x(t),t)) + B(t)(\gamma_2(x(t),t)) + B(t)u_1(t)\right],$$

Finally, to realize sliding mode, the relay control is designed

$$u_1(t) = -M(x(t), x(t-h), t)sign[s(t)], \tag{19}$$
$$M = q \parallel x(t) \parallel + \parallel x(t-h) \parallel) + p,$$
$$q > q_1, q_2, p > p_1$$

The next section presents the robustification of the designed optimal control (3). This robust regulator is designed assigning the sliding mode manifold according to (17)--(18) and subsequently moving to and along this manifold using relay control (19).

6 Robust Sliding Mode Control Design for Linear System with Time Delay in Control Input

Consider again the linear system (1) with time delay in control input, whose behavior is now affected by smooth uncertainties g_1, g_2 presenting perturbations and nonlinearities in the system (1)

$$dx(t) = (a_0(t) + a(t)x(t))dt + B(t)u(t-h)dt, \tag{20}$$

with the initial condition $x(s) = \varphi(s)$, $s \in [-h, 0]$, where $\varphi(s)$ is a piecewise continuous function given in the interval [-h,0]. It is also assumed that the disturbances satisfy the standard matching conditions

$$g_1(x(t),t) = B\gamma_1(x(t),t),$$
$$g_2(x(t-h),t) = B\gamma_2(x(t-h),t),$$
$$\parallel \gamma_1(x(t),t) \parallel \le q_1 \parallel x(t) \parallel + p_1, \qquad p_1, q_1 > 0,$$
$$\parallel \gamma_2(x(t-h),t) \parallel \le q_2 \parallel x(t-h) \parallel, \qquad q_2 > 0,$$

providing reasonable restrictions on their growth. The quadratic cost function (2) is the same as in Section 2.

uThe problem is to robustify the obtained optimal control (9), using the method specified by (17)--(18). Define this new control in the form (15): $u(t) = u_0(x(t-h)) + u_1(t)$, where the optimal control $u_0(x(t-h))$ coincides with (9) and the robstifying component $u_1(t)$ is obtained according to (19)

$$u_1(t) = -M(x(t), x(t-h), t) sign[s(t)],$$
$$M = q(\| x(t) \| + \| x(t-h) \|) + p,$$

$q > q_1, q_2, p > p_1$. Consequently, the sliding mode manifold function $s(t)$ is defined as

$$s(t) = z(t) + s_0(x(t)), \tag{21}$$

where

$$s_0 = u_0(x(t)) = R(t)^{-1} B^T \exp\left\{(-\int_{t-h}^t a^T(s)ds)\right\} Q(t)x(t-h), \tag{22}$$

and the auxiliary variable $z(t)$ satisfies the delay differential equation

$$\dot{z}(t) = -G(t)[a_0(t) + a(t)x(t) + B(t)u_0(x(t-h))] \tag{23}$$

with the initial conditions $z(\theta) = s_0(\varphi(\theta))$ for $\theta \in [-h, 0]$. In accordance with (18), the matrix $G(t)$ is equal to

$$G(t) = ds_0(x(t))/dx(t) = R(t)^{-1} B^T \exp\left\{(-\int_{t-h}^t a^T(s)ds)\right\} Q(t),$$

where $Q(t)$ is the solution of the Riccati equation (4).

7 Example

This section presents an example of designing the optimal regulator for a system (1) with a criterion (2) using the scheme (9)--(10), comparing it to the regulator where the matrix Q is selected as in the optimal linear regulator for a system without delays, disturbing the obtained optimal regulator by a noise, and designing a robust sliding mode compensator for that disturbance using the scheme (21)--(23).

Let us start with a scalar linear system

$$\dot{x}(t) = x(t) + u(t-0.1), \tag{24}$$

with the initial conditions $x(s) = 0$ for $s \in [-0.1, 0)$ and $x(0)=1$. The optimal control problem is to find the control $u(t)$, $t \in [0, T]$, T=0.25, that minimizes the criterion

$$J = \frac{1}{2}[x(T) - x^*]^2 + \frac{1}{2}\int_0^T u^2(t)dt, \tag{25}$$

where $T = 0.25$, and $x^* = 10$ is a large value of $x(t)$ *a priori* unreachable for time T. In other words, the optimal control problem is to maximize the state $x(t)$ using the minimum energy of control u.

Let us first construct the regulator where the optimal control law and the matrix $Q(t)$ are calculated in the same manner as for the optimal linear regulator for a linear system without delays in control input, that is $u_{opt}(t) = R(t)^{-1}B^T Q(t)x(t)$, (see [22] for reference). Since $B(t)=1$ in (24) and $R(t)=1$ in (25), the optimal control is actually equal to

$$u(t)= Q(t)x(t), \tag{26}$$

where $Q(t)$ satisfies the Riccati equation

$$\dot{Q}(t) = (-a^T(t)Q(t) - Q(t)a(t) + L(t) - Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T)= \psi$. Since $a(t)=1$, $B(t)=1$ in (24), and $L=0$ and $\psi=1$ in (25), the last equation turns to

$$\dot{Q}(t) = -2Q(t) - Q(t)^2, \qquad Q(0.25)=1. \tag{27}$$

Upon substituting the optimal control (26) into (24), the controlled system takes the form

$$\dot{x}(t) = x(t) + Q(t-0.1)x(t-0.1). \tag{28}$$

The results of applying the regulator (26), (27) to the system (24) are shown in Fig. 1, which presents the graphs of the controlled state (28) $x(t)$ in the interval $[0,T]$, the shifted ahead by 0.1 criterion (25) $J(t\text{-}0.1)$ in the interval $[0.1,T+0.1]$, and the shifted ahead by 0.1 control (26) $u(t\text{-}0.1)$ in the interval $[0,T]$. The values of the state (28) and the criterion (25) at the final moment $T=0.25$ are $x(0.25)=1.5097$ and $J(0.25)=36.2598$.

Let us now apply the optimal regulator (3)--(4) for linear systems with time delay in control input to the system (24). Since $a(t)=1$, $B(t)=1$, and $h=0.1$ in (24) and $=1$, $R(t)=1$,

and $L=0$ in (25), hence, $\exp\left\{(-\int_{t-h}^{t} a^T(s)ds)\right\} = \exp\{(0.1)\}$ and the optimal control law (9) takes the form

$$u_{opt}(t) = \exp\{(0.1)\}Q(t)x(t), \tag{29}$$

where $Q(t)$ satisfies the Riccati equation

$$\dot{Q}(t) = -2Q(t) - (\exp\{(0.1)\}Q(t))^2, \qquad Q(0.25)=1. \tag{30}$$

Upon substituting the optimal control (29) into (24), the optimally controlled system takes the form

$$\dot{x}(t) = x(t) + \exp\{(0.1)\}Q(t-0.1)x(t-0.1), \tag{31}$$

The results of applying the regulator (29),(30) to the system (24) are shown in Fig. 2, which presents the graphs of the optimally controlled state (31) $x(t)$ in the interval $[0,T]$, the shifted ahead by 0.1 criterion (25) $J(t\text{-}0.1)$ in the interval $[0.1,T+0.1]$, and the shifted ahead by 0.1 optimal control (29) $u_{opt}(t-0.1)$ in the interval $[0,T]$. The values of the state (31) and the criterion (29) at the final moment $T=0.25$ are $x(0.25)=1.668$ and $J(0.25)=35.3248$.

The next task is to introduce a disturbance into the optimally controlled system (31). This deterministic disturbance is realized as a constant: $g(t)$=-2. The matching conditions are valid, because state x(t) and control u(t) have the same dimension: dim(x)=dim(u)=1. The restrictions on the disturbance growth hold with $q_1 = 0$ and $p_1 = 2$, since $\|g(t)\| = 2$. The disturbed system equation (31) takes the form

$$\dot{x}(t) = -2 + x(t) + \exp\{(0.1)\}Q(t - 0.1)x(t - 0.1) \tag{32}$$

The system state behavior significantly deteriorates upon introducing the disturbance. Figure 3 presents the graphs of the disturbed state (32) x(t) in the interval [0,T], the shifted ahead by 0.1 criterion (25) J(t-0.1) in the interval [0.1,T+0.1], and the shifted ahead by 0.1 control (29) u(t-0.1) in the interval [0,T]. The values of the state (32) and the criterion (29) at the final moment T=0.25 are x(0.25)=1.0514 and J(0.25)=40.4596. The state (32) does almost not increase from its initial value x(0)=1, although it shouldbe maximized, and the criterion value does also almost not decrease and becomes much larger than in the preceding cases.

Let us finally design the robust integral sliding mode control compensating for the introduced disturbance. The new controlled state equation should be

$$\dot{x}(t) = -2 + x(t) + \exp\{(0.1)\}Q(t - 0.1)x(t - 0.1) + u_1(t), \tag{33}$$

where the compensator $u_1(t)$ is obtained according to (19)

$$u_1(t) = -M(x(t), x(t - h), t)sign[s(t)] \tag{34}$$

The sliding mode manifold s(t) is defined by (21)

$$s(t) = z(t) + s_0(x(t)),$$

where

$$s_0(x(t)) = u_0(x(t)) = \exp\{(0.1)\}Q(t)x(t),$$

and the auxiliary variable z(t) satisfies the delay differential equation

$$\dot{z}(t) = -G(t)[x(t) + u_0(x(t - h))] = -G(t)[x(t) + \exp\{(0.1)\}Q(t - 0.1)x(t - h)],$$

with the initial conditions $z(s) = 0$ for $s \in [-0.1, 0)$ and z(0)=-1. In accordance with (18), the matrix G(t) is equal to
$$G(t) = ds_0(x(t)) / dx(t) = \exp\{(0.1)\}Q(t),$$

where Q(t) is the solution of the Riccati equation (30).

Upon introducing the compensator (34) into the state equation (33), the system state behavior is very much improved. Figure 4 presents the graphs of the compensated state (33) x(t) in the interval [0,T], the shifted ahead by 0.1 criterion (25) J(t-0.1) in the interval [0.1,T+0.1], and the sum of the shifted ahead by 0.1 control (29) and the compensator (34), u(t-0.1)+$u_1(t)$, in the interval [0,0.25]. The values of the state (33) and the criterion (25) at the final moment T=0.25 are x(0.25)=1.6779 and J(0.25)=36.9207. Thus, the value of the controlled state after applying the compensator (34) is only insignificantly less than those values for the optimal regulator (29)—(30) for linear systems

with time delay in control input, and much better than the values for the disturbed system (32). Of course, the criterion value here is worse than for the optimal regulator (although it is 70 percent improved in comparison to the disturbed state), since an additional control energy is required to suppress the disturbance.

8 Conclusions

It was shown in this paper that, using the duality principle, the results obtained in [5]for optimal filters can be applied to the control problem given in (1). In order to robustify the solution against disturbances, integral sliding modes were used. The only requirement for the disturbances to be compensated is that they must hold the matching conditions given in section 5. An illustrative example was given where it can be clearly seen the system behavior under three relevant conditions for comparison: Using the optimal control law proposed without disturbances, the system including disturbances without compensator and the disturbed integral sliding mode compensated system. It could be seen that the state behavior with disturbances without compensator deteriorates compared to that without disturbances. Using the compensator, the state behaves exactly as that of the system without disturbance and, as expected, in order to deal with the disturbance, the control effort is bigger with the integral sliding mode compensator.

9 Appendix

Proof of the optimal control problem solution.Define the Hamiltonian funcion [29,19] for the optimal control problem (1), (2) as

$$H(x,u,q,t) = u^T R(t)u + x^T L(t)x + q^T \left[a_0(t) + a_1(t)x + B(t)u_1(u) \right],$$ (35)

where $u_1(u) = u(t-h)$. Applying the maximum principle condition $\partial H / \partial u = 0$ to this specific Hamiltonian function (35) yields

$\partial H / \partial u = 0 \Rightarrow R(t)u(t) + (\partial u_1(t) / \partial u)^T B^T(t)q(t) = 0.$

Upon denoting $(\partial u_1(t) / \partial u) = M$, the optimal control law is obtained as

$u^*(t) = R^{-1}(t)M^T B^T(t)q(t)$

Taking linearity and causality of the problem into account, let us seek q(t) as a linear function in x(t)

$$q(t) = -Q(t)x(t),$$ (36)

where Q(t) is a square symmetric matrix of dimension n. This yields the complete form of the optimal control

$$u^*(t) = R^{-1}(t)M^T B^T(t)Q(t)x(t).$$ (37)

Note that the transversality condition [29,19] for q(T) implies that $q(t) = -\partial J / \partial x(T) = -\psi x(T)$ and, therefore, Q(T)=ψ.
Using the co-state equation $dq(t) / dt = -\partial H / \partial x$, which gives

$$-dq(t) / dt = L(t)x(t) + a_1^T q(t),$$ (38)

and substituting (36) into (38), we obtain

$$\dot{Q} x(t) + Q(t)d(x(t)) / dt = L(t)x(t) - a_1^T Q(t)x(t). \tag{39}$$

Substituting the expression for $\dot{x}(t)$ from the state equation (1) into (39) yields

$$\dot{Q} x(t) + Q(t)a_1 x(t) + Q(t)B(t)u(t-h) = L(t)x(t) - a_1^T Q(t)x(t). \tag{40}$$

In view of linearity of the problem, differentiating the last expression in x does not imply loss of generality. Upon taking into account that $\partial u(t-h) / \partial x(t) = (\partial u(t-h) / \partial u(t))(\partial u(t) / \partial x(t)) = M(t)R^{-1}(t)M^T B^T(t)Q(t)$ and differentiating the equation (40) in x, it is transformed into the Riccati equation

$$\dot{Q} = L(t) - Q(t)a_1(t) - a_1^T Q(t) - Q(t)B(t)M(t)R^{-1}(t)M^T B^T(t)Q(t). \tag{41}$$

Let us find the value of matrix $M(t)$ for this problem. First of all, let us note [22] that the Hamiltonian function $H(x^*,u^*,q^*,t)$ is constant in t for the optimal control (37) $u^*(t)$, the corresponding optimal state (1) $x^*(t)$ and co-state $q^*(t)$ satisfying (36), and $Q(t)$ satisfying the equation (41), and equal to

$$H(x^*,u^*,q^*,t) = u^{*T} R(t)u^* + x^{*T} L(t)x^* + d(x^{*T} Q(t)x^*) / dt = C = \text{const}. \tag{42}$$

Integrating the last equality from t-h to t yields

$$\int_{t-h}^{t} \left[u^{*T}(s)R(s)u^*(s) + x^{*T}(s)L(s)x^*(s) \right] ds + x^{*T}(t)Q(t)x^*(t) - x^{*T}(t-h)Q(t-h)x^*(t-h) = Ch.$$

Differentiating the obtained formula respect to $x^*(t)$ and $u^*(t)$ and taking into account the optimal control expressions for $u^*(t)$ and $u^*(t$-$h)$ given by (37), we obtain

$$R^{-1}(t)M^T(t)B^T(t) = M^{-1}(t)R^{-1}(t-h)M^T(t-h)B^T(t-h)\exp\left\{ \int_{t-h}^{t} a^T(s)ds \right\}, \tag{43}$$

also using that

$$\partial x(t) / \partial x(t-h) = \exp\left\{ \int_{t-h}^{t} a^T(s)ds \right\}.$$

The last formula follows from the Cauchy formula for the solution of the linear state equation (1)

$$x(t) = \Phi(t,t-h)x(t-h) + \int_{t-h}^{t} \Phi(t,\tau)a_0(\tau)d\tau + \int_{t-h}^{t} \Phi(t,\tau)B(\tau)u(\tau-h)d\tau,$$

where $\Phi(t,\tau)$ is the matrix of fundamental solutions of the homogeneous equation (1), that is solution of the matrix equation

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$$\frac{d\Phi(t,\tau)}{dt} = a(t)\Phi(t,\tau), \qquad \Phi(t,t) = I,$$

where I is the identity matrix. In other words, $\Phi(t,t-h) = \exp\left\{\int_{t-h}^{t} a^T(s)ds\right\}$

Furthermore, it can be noted, differentiating twice the formula (42) with respect to x*(t), that the expression $R^{-1}(t)M^T(t)B^T(t)$ does actually not depend on $B(t)$ or $R^{-1}(t)$ as functions of time t. Thus, the value of the matrix $M(t)$ for this problem can be determined from (43) assuming that time t-h is equal to t in the matrix function $R^{-1}(t-h)M^T(t-h)B^T(t-h)$. Finally, the formula (43) gives the following equality for calculating $M(t)$

$$M^T(t)B^T(t) = B^T(t)\exp\left\{\int_{t-h}^{t} a^T(s)ds\right\} \tag{44}$$

Substituting the formula (44) into (37) and (41) yields the desired formulas (9) and (10) for the optimal control law $u*(t)$ and the matrix function $Q(t)$. The optimal control problem solution is proved.
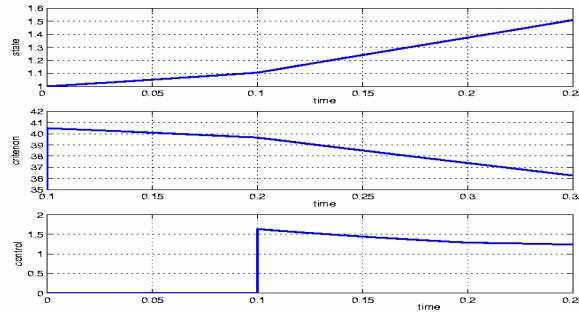

Fig. 1. Best linear regulator available for linear systems without delays. Graphs of the controlled state (28) $x(t)$ in the interval [0,0.25], the shifted ahead by 0.1 criterion (25) $J(t$-0.1) in the interval [0.1,0.35], and the shifted ahead by 0.1 control (26) $u(t$-0.1) in the interval [0,0.25].
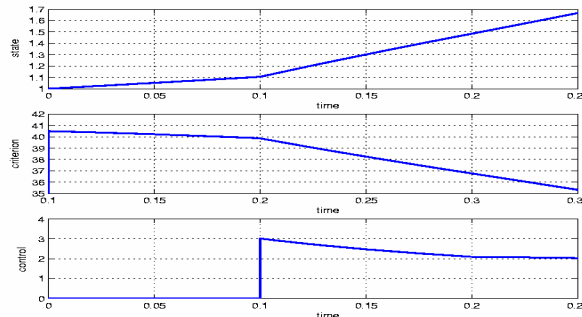
Fig. 2. Optimal regulator obtained for linear systems with time delay in control input. Graphs of the optimally controlled state (31) $x(t)$ in the interval [0,0.25], the shifted ahead by 0.1 criterion (25) $J(t$-0.1) in the interval [0.1,0.35], and the shifted ahead by 0.1 optimal control (29) $u_{opt}(t-0.1)$ in the interval [0,0.25].

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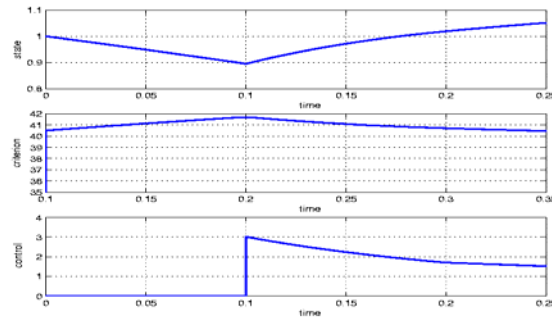
Fig. 3. Controlled system in the presence of disturbance. Graphs of the disturbed state (32) $x(t)$ in the interval [0,0.25], the shifted ahead by 0.1 criterion (25) $J(t\text{-}0.1)$ in the interval [0.1,0.35], and the shifted ahead by 0.1 control (29) $u(t\text{-}0.1)$in the interval [0,0.25].
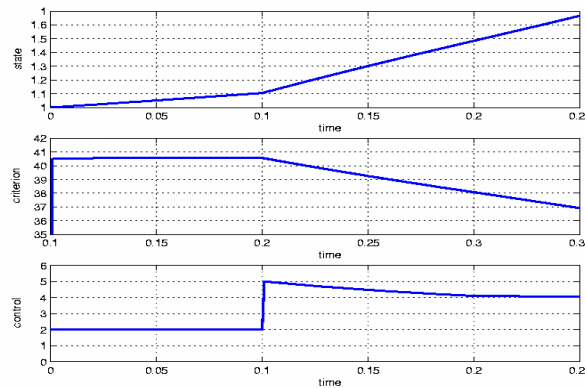
Fig. 4. Controlled system after applying robust integral sliding mode compensator. Graphs of the compensated state (33) $x(t)$ in the interval [0,0.25], the shifted ahead by 0.1 criterion (25) $J(t\text{-}0.1)$ in the interval [0.1,0.35], and the sum of the shifted head by 0.1 control (29) and the compensator (34), $u(t\text{-}0.1)+u_1(t)$ (equivalent sliding mode control), in the interval [0,0.25].

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