Stability Analysis of Proportional-Integral AQM Controllers Supporting TCP Flows

Análisis de Estabilidad de Controladores AQM Proporcional-Integral Suportando Flujos TCP

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Abstract
This paper focuses on the asymptotic stability of proportional-integral AQM controllers supporting TCP flows. Necessary and sufficient conditions for the asymptotic stability of the closed-loop linearization are provided. As a result, the complete set of proportional-integral controllers that locally stabilizes the equilibrium point is obtained. The robustness of the controllers to uncertainties in the network parameters (number of TCP flows, round-trip time and link capacity) is also addressed. It is shown that designing the controller with respect to the largest expected values of delay and link capacity, and the smallest expected value of TCP loads leads to the complete set of robust stabilizing controllers.
Keywords: Proportional-Integral AQM controllers, Stability, Robust Stability, Time-Delay Systems.

Resumen
Este artículo se centra en la estabilidad asintótica de controladores AQM del tipo proporcional-integral suportando flujos TCP . Se proporcionan condiciones necesarias y suficientes para la estabilidad asintótica de la linealización del sistema en lazo cerrado. Como un resultado, se obtiene el conjunto completo de controladores proporcional-integral que estabiliza localmente el punto de equilibrio. También se aborda la robustez de los controladores a incertidumbres en los parámetros de red (número de sesiones TCP, tiempo de viaje completo y capacidad del enlace). Se muestra que diseñando el controlador con respecto a los valores más grandes esperados para el retardo y la capacidad del enlace, y el valor más pequeño esperado para las cargas de TCP nos proporciona el conjunto completo de controladores robustos.
Palabras clave: Controladores AQM proporcional-integral; Estabilidad; Estabilidad Robusta; Sistemas con retardos.

1 Introduction

Recently, in (Misra, Gong and Towsley 2000) a fluid-flow model describing accurately the behavior of active queue management (AQM) supporting transmission control protocol (TCP) flows in congested routers was introduced. Such a model expresses AQM schemes as a classical feedback control problem. Motivated by the feedback control interpretation, proportional (P) and proportional-integral (PI) controllers have been proposed as AQM strategies in (Hollot, Misra, et al. 2002), and controllers based on the optimization of a H^∞ cost function in (Quet and Özbay 2004) for the linearized version of the model. It was shown there that such controllers improve the performance obtained with standard AQM controllers (e.g. based on Random Early Detection (RED)). The P and PI controller designs proposed by (Hollot, Misra, et al. 2002) are based on sufficient conditions for closed-loop stability of the linearization by focusing on the low-frequency dynamics and considering the high-frequency dynamics as an uncertainty. Such AQM schemes are currently implemented in Network Simulator (ns-2).

In the recent paper (Michiels, Melchor-Aguilar and Niculescu 2006) a linearized stability analysis of the simplified model introduced in (Hollot and Chait 2001) has been performed by taking a P controller as AQM strategy. It is shown there that necessary and sufficient conditions for asymptotic stability can be obtained by using frequency domain tools for the stability analysis of time-delay systems. Such a result provides the complete set of P controllers that locally stabilizes the equilibrium point. In addition, a nonlinear stability analysis has been developed and simple-to-check stability conditions in terms of the network parameters were derived by means of the Lyapunov-Krasovskii functional approach.

However, to the best of our knowledge, there are no specific results for the problem of finding the complete set of PI AQM controllers, and the aim of this paper is to focus in this problem. A first attempt to tackle this problem can be found in (Üstebay and Özbay 2006), where a new method for tuning the parameters of PI AQM controllers was proposed. The approach tries to find the largest allowable intervals for the controller's gains and select the "optimal gains" as the center of these intervals.

Recently in (Silva, Datta and Bhattacharyya 2005) a method of computing the complete set of PI controllers that stabilizes a certain class of linear time-invariant systems with time-delay has been proposed. Such a method is based on first determining the complete set of controller's gains that stabilizes the delay-free system for then to compute the controller's gains which make the system stable for delay values going from zero up to a minimal destabilizing delay. As we shall see in section 2, for TCP/AQM network systems it is not possible to investigate the stability for the delay-free case, and therefore the approach proposed in (Silva, Datta and Bhattacharyya 2005) can not be directly applied.

In this paper, we develop a local stability analysis of a simplified version of the model introduced in (Hollot, Misra, et al. 2002) for a PI control-based AQM strategy. Necessary and sufficient conditions for asymptotic stability of the linearized system are derived. More explicitly, for a given set of network parameters (round-trip time, number of TCP loads and link capacity) we obtain the complete set of PI controllers that locally stabilizes the equilibrium point. The robustness of the controllers to uncertainties in the network parameters is also addressed. We show that stabilizing with respect to the largest expected values of delay and link capacity, and the smallest expected value of TCP loads leads to the complete set of robust stabilizing controllers.

The remaining part of the paper is organized as follows: Section 2 introduces the mathematical model and controller design via linearization in a time-domain approach. The stability analysis is presented in section 3 and the robust stability analysis in section 4. A numerical simulation is presented in section 5. Several concluding remarks end the paper.

2 Mathematical model and controller design via linearization

We consider the dynamic fluid-flow model introduced in (Hollot, Misra, et al. 2002) for describing the behavior of a TCP/AQM network which ignores the time-out and slow start mechanisms. Such a model, relating the average values of key network variables of n homogeneous TCP-controlled sources and a single congested router, is described by the following coupled nonlinear differential equations with time-varying delay:

$$\begin{cases} \dot{w}(t) = \dfrac{1}{\tau(t)} - \dfrac{1}{2}\dfrac{w(t)w(t-\tau(t))}{\tau(t-\tau(t))}p(t-\tau(t)), \\ \dot{q}(t) = n(t)\dfrac{w(t)}{\tau(t)} - c, \end{cases}$$

where $w(t)$ denotes the average of TCP windows size (packets), $q(t)$ is the average queue length (packets), $\tau(t) = \dfrac{q(t)}{c} + \tau_p$ is the round-trip time (secs) where τ_p represents the propagation delay, c is the link capacity (packets/secs), n(t) is the number of TCP sessions and p( ) is the probability of packet marking. The first differential equation describes the TCP window control dynamics: the first term in the right-hand side corresponds to the window's additive-increase behavior, and the second term to the multiplicative-decrease behavior. The second equation describes the bottleneck queue length as the difference between the packet arrival rate $\dfrac{n(t)w(t)}{\tau(t)}$ and the link capacity c, assuming that there are no internal dynamics in the bottleneck.

Following (Hollot and Chait 2001), (Hollot, Misra, et al. 2002), (Michiels, Melchor-Aguilar and Niculescu 2006) and (Üstebay and Özbay 2006) we assume that $n(t) \equiv n$ and $\tau(t) \equiv \tau$ are constants. The assumption on the delay can be considered as a good approximation when the queuing delay is much smaller than the propagation

delay, which occurs when the link capacity c is sufficiently large. As a result, we obtain the following simplified dynamics:

$$
\begin{cases}
\dot{w}(t) = \dfrac{1}{\tau} - \dfrac{1}{2\tau}w(t)w(t - \tau)p(t - \tau), \\
\dot{q}(t) = \dfrac{n}{\tau}w(t) - c.
\end{cases}
\tag{1}
$$

Taking (w, q) as the state and p as the input, the equilibrium point (w_0, q_0, p_0) is defined by

$$
w_0^2 p_0 = 2 \quad and \quad w_0 = \frac{\tau c}{n}.
$$

In (Hollot and Chait 2001) it was argued that if $w_0 \gg 1$, then the local behavior of (1) about the equilibrium can be approximated by the local behavior of

$$
\begin{cases}
\dot{w}(t) = \dfrac{1}{\tau} - \dfrac{1}{2\tau}w^2(t)p(t - \tau), \\
\dot{q}(t) = \dfrac{n}{\tau}w(t) - c,
\end{cases}
\tag{2}
$$

having the same equilibrium, (Hollot and Chait 2001), and (Michiels, Melchor-Aguilar and Niculescu 2006) for a mathematical justification. The condition $w_0 \gg 1$ is a reasonable assumption for typical network parameters, where the link capacity is sufficiently large to consider the delay value as constant. Hence, (2) is an acceptable model to investigate the local behavior of some classes of TCP/AQM networks described by (1).

In order to design a stabilizing PI controller via the linearization of the system about the equilibrium point, we define

$$
\sigma(t) = \int_0^\tau (q(s) - q_0)ds,
$$

and consider the augmented system

$$
\begin{cases}
\dot{w}(t) = \dfrac{1}{\tau} - \dfrac{1}{2\tau}w^2(t)p(t - \tau), \\
\dot{q}(t) = \dfrac{n}{\tau}w(t) - c,
\end{cases}
\tag{3}
$$

The control p will be designed as a function of (q, σ) such that the closed-loop system has an equilibrium (w_0, q_0, σ_0), where σ_0 will be designed by feedback. The linearization of (3) about the equilibrium $(w_0, q_0, p_0, \sigma_0)$ is

$$
\dot{\xi}(t) = A\xi(t) + b\tilde{p}(t - \tau),
\tag{4}
$$

where $\xi(t) = \begin{pmatrix} \tilde{w}(t) \\ \tilde{q}(t) \\ \tilde{\sigma}(t) \end{pmatrix}$, $A = \begin{pmatrix} -\frac{2n}{\tau^2 c} & 0 & 0 \\ \frac{n}{\tau} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} -\frac{\tau^2 c^2}{2n^2} \\ 0 \\ 0 \end{pmatrix}$, $\tilde{w}(t) = w(t) - w_0$, $\tilde{q}(t) = q(t) - q_0$,

$\tilde{\sigma}(t) = \sigma(t) - \sigma_0$, and $\tilde{p}(t) = p(t) - p_0$. Now we consider a PI controller of the form

$$p(t) = k_p \tilde{q}(t) + \frac{k_p}{l} \tilde{\sigma}(t),\tag{5}$$

where $\frac{k_p}{l} \neq 0$. The closed-loop system (4)-(5) has the following form:

$$\dot{\xi}(t) = A\xi(t) + B\xi(t - \tau),\tag{6}$$

where

$$B = \begin{pmatrix} 0 & -\frac{\tau^2 c^2}{2n^2} k_p & -\frac{\tau^2 c^2 k_p}{2n^2\, l} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Assuming for the moment that it is possible to find controller's gains that make (6) asymptotically stable, the controller should be taken as

$$p(t) = k_p(q(t) - q_0) + \frac{k_p}{l}(\sigma(t) - \sigma_0) + p_0.$$

Choosing σ_0 as $\sigma_0 = \frac{l}{k_p}(p_0 - k_p q_0)$ the controller simplifies to

$$p(t) = k_p q(t) + \frac{k_p}{l} \sigma(t).\tag{7}$$

It can be easily checked that the closed-loop system (3)-(7) has a unique equilibrium point (w_0, q_0, σ_0), and all solutions of (3) starting sufficiently close to it approach it as t tends to infinity.

Remark 1: It is clear that one can not investigate the stability of (6) for the delay-free case $(\tau = 0)$. This is a particular property of the TCP/AQM network systems, where the delay value (round-trip time) can not be considered zero. Thus, the approach developed in (Silva, Datta and Bhattacharyya 2005) can not be directly applied to determine the complete set of PI stabilizing controllers for (6).

3 Stability

It is well known that (6) is asymptotically stable if and only if the characteristic function

$$f(s) = s^3 + \frac{2n}{\tau^2 c}s^2 + \frac{c^2}{2n}e^{-\tau s}k_p\left(s + \frac{1}{l}\right)$$

has no zeros with nonnegative real parts (Gu, Kharitonov and Chen 2003), (Niculescu 2001).
Necessary and sufficient stability conditions are expressed in the following theorem.

Theorem 2: *System (6) is exponentially stable if and only if the controller's gains (k_p, I) belong to the stability region $\Phi_{(n,\tau,c)}$, plotted in Fig. 1, whose boundary in the controller's gains space (k_p, I)) is described by*

$$\partial\Phi_{(n,\tau,c)} = \left\{ (k_p, I) : I = \frac{\omega\cos(\omega\tau) + \frac{2n}{\tau^2 c}\sin(\omega\tau)}{\omega\left(\frac{2n}{\tau^2 c}\cos(\omega\tau) - \omega\sin(\omega\tau)\right)}, \; k_p = \frac{2n}{c^2}\omega\left(\cos(\omega\tau) + \frac{2n}{\tau^2 c}\sin(\omega\tau)\right), \; \omega \in (0, \omega^*) \right\}$$

where ω^* is the solution of the equation

$$\tan(\omega\tau) = \frac{2n}{\tau^2 c\omega}, \; \omega \in (0, \omega^*),$$

Proof: First observe that since $\frac{k_p}{I} \neq 0, s = 0$, is not a zero of $f(s)$. Suppose that $f(s)$ has a pure imaginary zero $s = j\omega \neq 0$. Then, a direct calculation yields to

$$\begin{cases} k_p = \frac{2n}{c^2}\omega\left(\cos(\omega\tau) + \frac{2n}{\tau^2 c}\sin(\omega\tau)\right) \\[2mm] I = \frac{\omega\cos(\omega\tau) + \frac{2n}{\tau^2 c}\sin(\omega\tau)}{\omega\left(\frac{2n}{\tau^2 c}\cos(\omega\tau) - \omega\sin(\omega\tau)\right)} \end{cases} . \tag{8}$$

This parametrization defines a countable number of curves in the parameter space (k_p, I) and each one of them is obtained by varying ω in an interval $(\omega_k^2, \omega_{k+1}^2), k = 0,1,2,...,$ where ω_k^2 is the solution of the equation

$$\tan(\omega\tau) = \frac{2n}{\tau^2 c\omega}, \; \omega \in \left(\frac{k\pi}{\tau}, \frac{(2k+1)\pi}{2\tau}\right), k = 0,1,2,.... \tag{9}$$

Since (9) is a transcendental equation we look directly for a numerical solution. This can be found by plotting the two functions $\frac{2n}{\tau^2 c\omega}$ and $\tan(\omega\tau)$, see Fig. 2. These curves partition the plane (k_p, I) into a set of connected domains. From the argument principle is easy to show that for all (k_p, I) values inside the open domain $\Phi_{(n,\tau,c)}$, bounded by the curve for $k = 0$ and the coordinate axis $k_p = 0$, the function $f(s)$ has no roots with strictly positive real part.

Remark 3: *From parametrization (8) it is not difficult to see that $I(\omega) \rightarrow \frac{\tau c}{2n} + \tau$ when $\omega\rightarrow+0$ and $I(\omega)\rightarrow+\infty$ when $\omega \rightarrow -\omega^*$. On the other hand, it holds that $k_p(\omega) \rightarrow k_p(0) = 0$ when $\omega\rightarrow+0$ and $k_p(\omega) \rightarrow k_p(\omega^*) = \frac{2n(\omega^*)^2}{c\cos(\omega^*\tau)}$ when $\omega \rightarrow -\omega^*$.*

We say that a scalar function $g(\omega)$ defined in an interval (ω_a, ω_b) has a critical point at $\omega_0 \in (\omega_a, \omega_b)$ if $g'(\omega_0) = 0$.

Lemma 4: *The function $k_p(\omega)$ defined by parameterization (8) has at most one critical point in the interval $\left(0, \frac{\pi}{2\tau}\right)$. The critical point, if it exists, is a maximum.*
Proof: Let us compute

$$k_p'(\omega) = \frac{2n}{c^2}\left[\left(2\omega + \frac{2n}{\tau c}\omega\right)\cos(\omega\tau) + \left(\frac{2n}{\tau^2 c} - \tau\omega^2\right)\sin(\omega\tau)\right].$$

The zeros of $k_p'(\omega)$ in the interval $\left(0, \frac{\pi}{2\tau}\right)$ are the solutions of the equation

$$m(\omega) = \tan(\omega\tau), \omega \in \left(0, \frac{\pi}{2\tau}\right), \tag{10}$$

where

$$m(\omega) = \frac{\left(2 + \frac{2n}{\tau c}\right)\omega}{\left(\tau\omega^2 - \frac{2n}{\tau^2 c}\right)}.$$

Again, since (10) is a transcendental equation we look directly for a numerical solution by plotting the functions $m(\omega)$ and $\tan(\omega\tau)$, see Fig. 3. The function $m(\omega)$ is undetermined at $\bar{\omega} = \frac{1}{\tau}\sqrt{\frac{2n}{\tau c}}$. It is clear that if $\bar{\omega} \geq \frac{\pi}{2\tau}$ there is no solution of (10) and if $\bar{\omega} < \frac{\pi}{2\tau}$ there is always a solution of (10), which implies that there is a unique critical point of $k_p(\omega)$ in the interval $\left(0, \frac{\pi}{2\tau}\right)$. Thus, the first part of the lemma is proved. It remains to prove that such critical point, when there exists, is a maximum. Let $\bar{\omega}$ be the solution of (10). From Fig. 3 we have the following inequalities:

1. $m(\omega) > \tan(\omega\tau), \forall\omega \in (\bar{\omega}, \bar{\omega})$. Since $\cos(\omega\tau) > 0$ and $\tau\omega^2 - \frac{2n}{\tau^2 c} > 0$ for all $\omega \in (\bar{\omega}, \bar{\omega})$ it follows that $k_p'(\omega) > 0, \forall\omega \in (\bar{\omega}, \bar{\omega})$.

2. $m(\omega) < \tan(\omega\tau), \forall\omega \in \left(\bar{\omega}, \frac{\pi}{2\tau}\right)$. Since $\cos(\omega\tau) > 0$ and $\tau\omega^2 - \frac{2n}{\tau^2 c} > 0$ for all $\omega \in \left(\bar{\omega}, \frac{\pi}{2\tau}\right)$ it follows that $k_p'(\omega) < 0, \forall\omega \in \left(\bar{\omega}, \frac{\pi}{2\tau}\right)$.

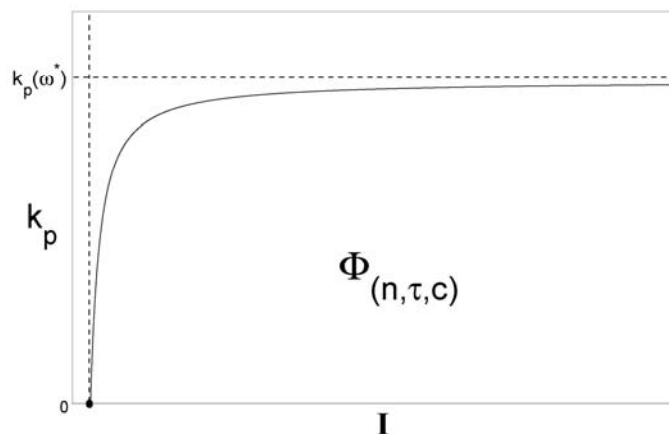Thus $k_p(\bar{\omega})$ is a maximum.

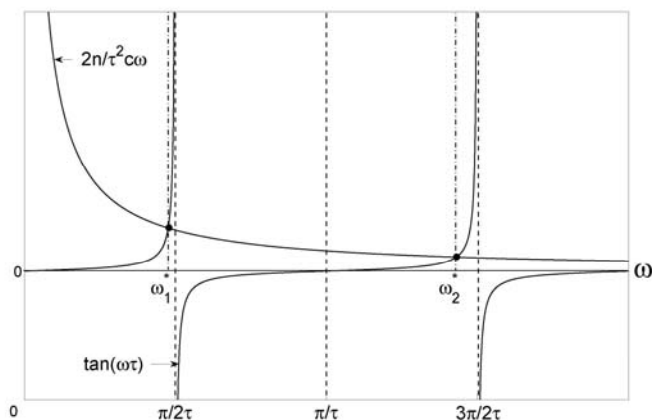**Fig. 1** Stability región $\Phi_{(n,\tau,c)}$

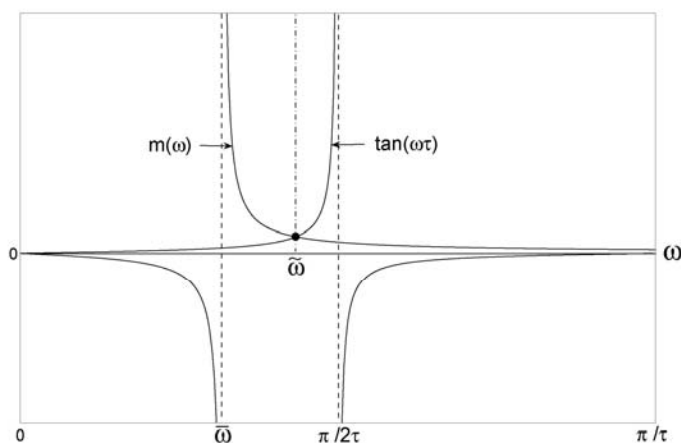**Fig. 2** Numerical solution of (9)

Fig. 3 Numerical solution of (10)

Lemma 5: *The function $k_2(\omega)$ defined by parametrization (8) is a monotonically increasing function of ω in the interval $(0, \omega^*)$.*

Proof: In virtue of lemma 4 it is sufficient to prove that for any given network parameters (n, τ, c) it holds that $\omega^* < \bar{\omega}$. First observe that if for some network parameters (n, τ, c) it holds that $\bar{\omega} \geq \omega^*$ then $\omega^* < \bar{\omega}$. Based on the continuity of the functions $\frac{2n}{\tau^2 c}$, $m(\omega)$ and $\tan(\omega\tau)$ w.r.t. the network parameters let us assume that there are network parameters (n, τ, c) such that $\bar{\omega} < \omega^*$ and $\omega^* = \bar{\omega}$. From (9) and (10) we have
and therefore holds.

$$2n\tau\bar{\omega}^2 - \frac{4n^2}{\tau^2 c} = 2\tau^2 c\bar{\omega}^2 + 2n\tau\bar{\omega}^2$$

This implies that there are no network parameters (n, τ, c) such that both $\bar{\omega} < \omega^*$ and $\omega^* = \bar{\omega}$ hold. The contradiction proves the lemma.

Lemma 6: *The function $I(\omega)$ defined by parameterization (8) is a monotonically increasing function of ω in the interval $(0, \omega^*)$.*

Proof: Let $I'(\omega) = \frac{N(\omega)}{D(\omega)}$ and $a = \frac{cn}{n^2 \tau}$. Then

$$N(\omega) = \omega \cos(\omega\tau)\left[a\cos(\omega\tau) - \omega\sin(\omega\tau)\right] + \omega\tau[a^2 + \omega^2] \tag{11}$$
$$- [\omega\cos(\omega\tau) + a\sin(\omega\tau)][a\cos(\omega\tau) - 2\omega\sin(\omega\tau)]$$

and $D(\omega) = \omega^2[a\cos(\omega\tau) - \omega\sin(\omega\tau)]^2$. The third term of the right-hand side of (11) can be written as

$$-[\omega\cos(\omega\tau) + a\sin(\omega\tau)][a\cos(\omega\tau) - 2\omega\sin(\omega\tau)]$$
$$= -[\omega\cos(\omega\tau) + a\sin(\omega\tau)][a\cos(\omega\tau) - \omega\sin(\omega\tau)] + \omega\sin(\omega\tau)[\omega\cos(\omega\tau) + a\sin(\omega\tau)]$$

Taking into account this equality in (11) we can easily arrive at the following expression:

$$N(\omega) = \frac{a^2}{2}[2\omega\tau - \sin(\omega\tau)] + a\omega\sin(\omega\tau)^2 + \omega^3\tau + \omega\sin(\omega\tau)[\omega\cos(\omega\tau) + a\sin(\omega\tau)].$$

Since $\omega\cos(\omega\tau) + a\sin(\omega\tau) > 0$ and $\sin(\omega\tau) > 0, \omega \in (0, \omega^*)$, and $2\omega\tau - \sin(\omega\tau) > 0, \omega \in \mathbb{R}_+$, it follows $N(\omega) > 0, \forall\omega \in (0, \omega^*)$, which implies that $I'(\omega) > 0, \forall\omega \in (0, \omega^*)$.

4 Robust Stability

In this section we address the robustness problem of the controller to uncertainties in the network parameters. Let us consider nominal network parameters (n_0, τ_0, c_0) and unknown network parameters (n, τ, c) satisfying

$$0 < n_0 \leq n, 0 < \tau \leq \tau_0 \text{ and } 0 < c \leq c_0. \tag{12}$$

The aim of this section is to determine the complete set of PI controllers which locally stabilizes the equilibrium point of the closed-loop system (3)-(7) for all network parameters (n, τ, c) satisfying (12). In the sequel, we will denote $k_p(\omega, n, \tau, c)$ and $I(\omega, n, \tau, c)$ the functions defined by parametrization (8) in order to emphasize the dependence on the network parameters.

Let ω_0^* and ω^* be the solutions of (9) corresponding to the network parameters (n_0, τ_0, c_0) and (n, τ, c) respectively.

Remark 7: *For network parameters (n, τ, c) and (n_0, τ_0, c_0) satisfying (12), $\omega_0^* \leq \omega^*$ holds.*

Proof: The remark can be shown to be true by plotting the functions involved in (9) corresponding to the network parameters (n_0, τ_0, c_0) and (n, τ, c) respectively.

Let $k_p(\omega, n_0, \tau_0, c_0)$ and $k_p(\omega, n, \tau, c)$ be the functions defined by parameterization (8) corresponding to the network parameters (n_0, τ_0, c_0) and (n, τ, c) respectively.

Lemma 8: *For network parameters (n, τ, c) and (n_0, τ_0, c_0) satisfying (12) the following inequality holds:*
$$k_p(\omega, n, \tau, c) \geq k_p(\omega, n_0, \tau_0, c_0), \forall\omega \in [0, \omega_0^*].$$

Proof: Let us compute the derivative of $k_p(\omega, n, \tau, c)$ w.r.t. the parameter τ. We have

$$\frac{\partial k_p(\omega, n, \tau, c)}{\partial \tau} = \frac{2n}{c^2}\omega\left[\frac{2n}{\tau^2 c}\omega\cos(\omega\tau) - \left(\omega^2 + \frac{4n}{\tau^2 c}\right)\sin(\omega\tau)\right],$$

Consider the following function

$$\zeta(\omega) = \frac{\frac{2n}{\tau^2 c}\omega}{\omega^2 + \frac{4n}{\tau^2 c}}.$$

The derivative of $\zeta(\omega)$ is

$$\zeta'(\omega) = \frac{\frac{2n}{\tau^2 c}\left(\frac{4n}{\tau^2 c} - \omega^2\right)}{\left(\omega^2 + \frac{4n}{\tau^2 c}\right)^2}. \tag{13}$$

The function $\zeta(\omega)$ has a unique maximum at $\bar{\omega} = \frac{2}{\tau}\sqrt{\frac{n}{\tau c}}$. Thus, the function $\zeta(\omega)$ is monotonically increasing for $\omega \in (0, \bar{\omega}]$ and monotonically decreasing for $\omega \in (\bar{\omega}, +\infty)$.

From (13) we have

$$\zeta'(\omega) < \frac{\frac{2n}{\tau^2 c}\left(\frac{4n}{\tau^2 c}\right)}{\left(\frac{4n}{\tau^2 c}\right)^2} = \frac{\tau}{2}, \forall\omega \in (0, \bar{\omega}),$$

which implies that $\zeta(\omega) < \tau\omega, \forall\omega \in (0, \bar{\omega})$. Then, $\zeta(\omega) < \tan(\tau\omega), \forall\omega \in \left(0, \frac{\pi}{2\tau}\right)$ holds. Since $\cos(\omega\tau) > 0, \forall\omega \in \left(0, \frac{\pi}{2\tau}\right)$ it follows that

$$\frac{2n}{\tau^2 c}\omega\cos(\omega\tau) < \left(\omega^2 + \frac{4n}{\tau^2 c}\right)\sin(\omega\tau),$$

which implies $\frac{\partial}{\partial\tau}k_p(\omega, n, \tau, c) < 0, \forall\omega \in \left(0, \frac{\pi}{2\tau}\right)$. Hence, $k_p(\omega, n, \tau, c)$ is a monotonically decreasing function of τ for all $\omega \in (0, \omega_0^*]$.

By computing the derivative of $k_p(\omega, n, \tau, c)$ w.r.t n we obtain

$$\frac{\partial}{\partial n}k_p(\omega, n, \tau, c) = \frac{2\omega}{c^2}\left[\omega\cos(\omega\tau) + \frac{2n}{\tau^2 c}\sin(\omega\tau)\right] + \frac{4n}{\tau^2 c^3}\sin(\omega\tau) > 0, \forall\omega \in (0, \omega_0^*]$$

and w.r.t c we have

$$\frac{\delta}{\partial c}k_p(\omega,n,\tau,c) = -\frac{4n\omega}{c^3}\left[\omega\cos(\omega\tau)+\frac{2n}{\tau^2 c}\sin(\omega\tau)\right]-\frac{2n^2}{\tau^2 c^4}\sin(\omega\tau) < 0,\forall\omega \in (0,\omega_0^*].$$

Thus, $k_p(\omega,n,\tau,c)$ is a monotonically increasing function of n and a monotonically decreasing function of c for all $\omega \in (0,\omega_0^*]$. Observing that $k_p(0,n,\tau,c) = k_p(0,n_0,\tau_0,c_0) = 0$ we conclude that for network parameters (n,τ,c) and (n_0,τ_0,c_0) satisfying (12) the lemma follows.

Consider now the functions $I(\omega,n_0,\tau_0,c_0)$ and $I(\omega,n,\tau,c)$ defined by parametrization (8) associated to the network parameters (n_0,τ_0,c_0) and (n,τ,c) respectively.

Lemma 9: *For network parameters (n,τ,c) and (n_0,τ_0,c_0) satisfying (12) it holds that*

$$\frac{k_p(\omega,n,\tau,c)}{I(\omega,n,\tau,c)} \geq \frac{k_p(\omega,n_0,\tau_0,c_0)}{I(\omega,n_0,\tau_0,c_0)},\forall\omega \in (0,\omega_0^*). \tag{14}$$

Proof: From the parameterization (8) we have

$$\frac{k_p(\omega,n,\tau,c)}{I(\omega,n,\tau,c)} = \frac{2n}{c^2}\omega^2\left[\frac{2n}{\tau^2 c}\cos(\omega\tau)-\omega\sin(\omega\tau)\right],$$

that can be written as

$$\frac{k_p(\omega,n,\tau,c)}{I(\omega,n,\tau,c)} = \frac{2n}{c^2}\omega^2\cos(\omega\tau)\left[\frac{2n}{\tau^2 c\omega}-\tan(\omega\tau)\right].$$

By plotting the functions $\frac{2n}{\tau^2 c\omega}$ and $\tan(\omega\tau)$ corresponding to the network parameters (n,τ,c) and (n_0,τ_0,c_0) we obtain the following inequalities:

$$\tan(\omega\tau) \leq \tan(\omega\tau_0),\forall\omega \in \left(0,\frac{\pi}{2\tau_0}\right),$$

$$\frac{2n}{\tau^2 c\omega} \geq \frac{2n_0}{\tau_0^2 c_0\omega},\forall\omega > 0.$$

It follows that

$$\frac{k_p(\omega,n,\tau,c)}{I(\omega,n,\tau,c)} \geq \frac{2n}{c^2}\omega^2\cos(\omega\tau)\left[\frac{2n_0}{\tau_0^2 c_0\omega}-\tan(\omega\tau_0)\right],\forall\omega \in (0,\omega_0^*).$$

Since $\cos(\omega\tau) > \cos(\omega\tau_0),\forall\omega \in \left(0,\frac{\pi}{2\tau_0}\right)$ and $\frac{2n}{c^2} \geq \frac{2n_0}{c_0^2}$ we get

$$\frac{k_p(\omega,n,\tau,c)}{I(\omega,n,\tau,c)} \geq \frac{2n_0}{c_0^2}\omega^2\cos(\omega\tau_0)\left[\frac{2n_0}{\tau_0^2 c_0\omega}-\tan(\omega\tau_0)\right],\forall\omega \in (0,\omega_0^*).$$

which implies

$$\frac{k_p(\omega, n, \tau, c)}{I(\omega, n, \tau, c)} \geq \frac{k_p(\omega, n_0, \tau_0, c_0)}{I(\omega, n_0, \tau_0, c_0)}, \forall \omega \in (0, \omega_0^c).$$

Theorem 10: *Given network parameters* (n, τ, c) *and* (n_0, τ_0, c_0) *satisfying (12) it holds that*

$$\Phi_{(n_0, \tau_0, c_0)} \subseteq \Phi_{(n, \tau, c)}.$$

Proof: Obviously, $\Phi_{(n, \tau, c)} = \Phi_{(n_0, \tau_0, c_0)}$ when $n = n_0, \tau = \tau_0$ and $c = c_0$. Thus, let us consider the non trivial case when $n \gg n_0, \tau \ll \tau_0$ and $c \ll c_0$. First we observe that $\partial \Phi_{(n_0, \tau_0, c_0)}$ and $\partial \Phi_{(n, \tau, c)}$ are determined by sweeping ω in the intervals $(0, \omega_0^c)$ and $(0, \omega^*)$ in the parametrization (8). The lemma 9 implies that by sweeping ω in the interval $(0, \omega_0^c)$, the stability region $\Phi_{(n_0, \tau_0, c_0)}$ is partially contained in the stability region $\Phi_{(n, \tau, c)}$. Thus, it remains to prove that by sweeping ω in the interval $[\omega_0^c, \omega^*)$ the corresponding part of $\partial \Phi_{(n, \tau, c)}$ does not intersect $\partial \Phi_{(n_0, \tau_0, c_0)}$.

From lemma 5 we have that the function $k_p(\omega, n, \tau, c)$ monotonically increases from $k_p(\omega_0^c, n, \tau, c)$ to $k_p(\omega^*, n, \tau, c) > k_p(\omega_0^c, n_0, \tau_0, c_0)$ when ω goes from ω_0^c to ω^*. From lemma 6 it follows that $I(\omega, n_0, \tau_0, c_0)$ is a monotonically increasing function of ω in the interval $[\omega_0^c, \omega^*)$ and from lemma 8 we have that $k_p(\omega_0^c, n, \tau, c) \geq k_p(\omega_0^c, n_0, \tau_0, c_0)$. Then we conclude that $\Phi_{(n_0, \tau_0, c_0)} \subseteq \Phi_{(n, \tau, c)}$.

Corollary 11: Assume that a controller (7) locally stabilizes the equilibrium point of (3) with network parameters (n_0, τ_0, c_0). Then it locally stabilizes the equilibrium point of (3) with network parameters (n, τ, c) satisfying (12).

Remark 12: *Corollary is similar to Proposition 2 in* (Hollot, Misra, et al. 2002)*, which states that designing the controller for the largest expected values of τ and c, and the smallest expected value of n yields a robustly stabilizing controller. The main different with* (Hollot, Misra, et al. 2002) *lies in the fact that with our approach the controller's gains can be selected based on the exact stability region, not on an estimate of it.*

Remark 13: *Assume that the network parameters* (n, τ, c) *are constants satisfying*

$$n \in [n_L, n_g], \tau \in [\tau_L, \tau_g] \text{ and } c \in [c_L, c_g]. \tag{15}$$

Then from theorem 10 it follows that $\Phi_{(n_L, \tau_g, c_g)} \subseteq \Phi_{(n, \tau, c)}$ *for all network parameters* (n, τ, c) *satisfying (15). Thus, by designing the controller (7) to locally stabilize the equilibrium point of (3) with network parameters* (n_L, τ_g, c_g) *yields a robustly stabilizing controller for all network parameters* (n, τ, c) *satisfying (15).*

5 Simulations

Consider the following nominal values for network parameters: $n_0 = 40$ TCP sessions, $c_0 = 300$ packets/secs and $\tau_0 = 0.7$ secs. As real network parameters consider the following values: $n = 50$ TCP sessions, $c = 300$ packets/secs and $\tau = 0.333$ secs. The operation point was chosen as $q_0 = 600$ packets with initial condition $q(0) = 300$ packets, i.e., under a no empty initial queue length (300 packets) in the router the objective is to regulate at 600 packets.

From corollary 11 it follows that designing the PI controller for the network parameters (n_0, τ_0, c_0) yields a robustly stabilizing controller. In Fig.4 we plot the stability region $\Phi_{(n_0, \tau_0, c_0)}$ in the controller's gains space. In order to illustrate some of the advantages of having the complete stability region, let us select two pairs of gains (k_p, I) inside of $\Phi_{(n_0, \tau_0, c_0)}$: $a = (1 \times 10^{-4}, 5)$ and $b = (3.30 \times 10^{-4}, 5)$, see Fig 4.

   In Fig. 5 we plot the responses of q(t) for both a and b gains. The simulations were carried out on the nonlinear model (3) by using Matlab/Simulink. It can be seen that the robust stabilization is reached. On the other hand, the responses obtained for the two different pairs of gains show the importance of knowing the complete set of controller parameter values that locally stabilizes the closed-loop system.
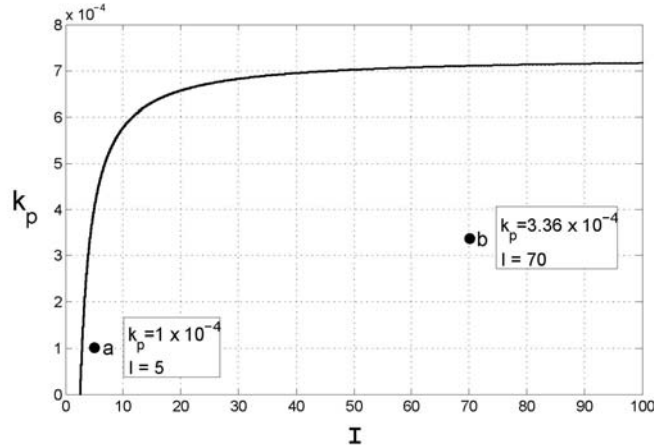

Fig. 4 Stability region $\Phi_{(n_0, \tau_0, c_0)}$ for $n_0 = 40$, $\tau_0 = 0.7$ and $c_0 = 300$
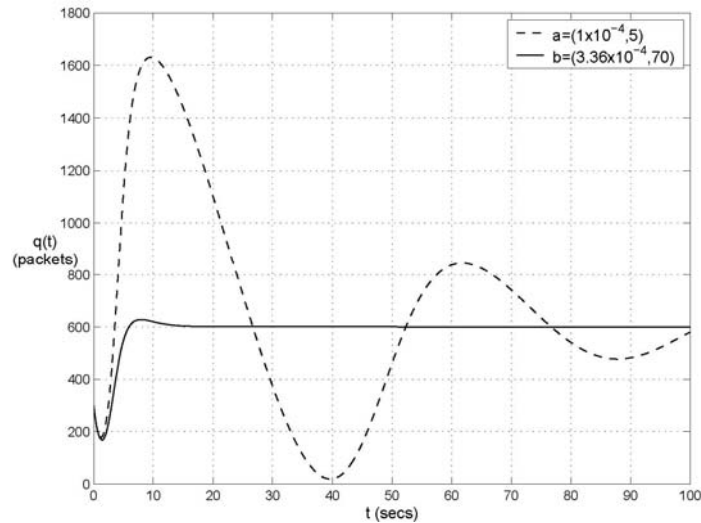

Fig. 5 Response of $q(t)$ for the a and b gains

6 Conclusions

In this paper we developed a linearized stability analysis of a fluid-flow model describing some classes of TCP/AQM networks. We consider a proportional-integral controller as AQM strategy. Necessary and sufficient conditions for asymptotic stability of the closed-loop linearization were obtained. This result provides the complete set of proportional-integral AQM controllers that locally stabilizes the equilibrium point in counterpart with the existing works in the literature which give only an estimate of this set.

The complete set of robust stabilizing controllers is also obtained. Thus, we showed that stabilizing with respect to the largest expect values of delay and link capacity, and the smallest expect value of TCP loads provides the complete set of robust stabilizing controllers.

We performed nonlinear simulations using Matlab/Simulink that illustrate the capabilities of the results towards further analysis oriented to determine some performance objectives.

7 Acknowledgments

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