

# Control of Mechanical Systems with Dry Friction

Roque Martínez<sup>1</sup> and Joaquín Álvarez<sup>2</sup>

<sup>1</sup> Programa de Ingeniería Mecánica, Unidad Académica de Ingeniería,  
Universidad Autónoma de Zacatecas, Zacatecas, Zac.,  
Mexico

<sup>2</sup> Departamento de Electrónica y Telecomunicaciones,  
División de Física Aplicada, CICESE, Ensenada, B. C.,  
Mexico

rmartinez@uaz.edu.mx, jqalvar@cicese.mx

**Abstract.** A technique to design a dynamic continuous controller to regulate a class of full-actuated mechanical systems with dry friction is proposed. It is shown that the control eliminates the steady-state error and is robust with respect to parameter uncertainties. A simple method to find the parameters of the controller is also proposed. Moreover, an application of this result to control a 2-DOF underactuated mechanical system with dry friction in the non-actuated joint is described. Here, the control objective is to regulate the non-actuated variable while the position and speed of the actuated joint remain bounded. Performance issues of the developed synthesis are illustrated with numerical and experimental results.

**Keywords.** Stability, friction, mechanical systems, underactuated systems.

## Control de sistemas mecánicos con fricción seca

**Resumen.** Se propone una estrategia de diseño de un controlador dinámico continuo para regular una clase de sistemas mecánicos totalmente actuados con fricción seca. Se demuestra que el control elimina el error en estado estacionario y que es robusto frente a cierto tipo de incertidumbres en los parámetros del sistema. Se propone también un método sencillo para calcular los parámetros del controlador. Además, se describe la aplicación de este resultado al control de sistemas subactuados de 2 grados de libertad, con fricción seca en la articulación no actuada. En este caso, el objetivo de control es regular la variable no actuada, manteniendo limitadas las amplitudes de la posición y de la velocidad de la articulación actuada. El desempeño del controlador propuesto se ilustra con resultados numéricos y experimentales.

**Palabras clave.** Estabilidad, fricción, sistemas mecánicos, sistemas subactuados.

## 1 Introduction

Dry friction is defined as a force that resists relative motion between contacting surfaces of different bodies. The bodies “stick” when the relative velocity between the contacting surfaces is zero. If the bodies slide over each other with a non-zero velocity, we speak of a “slip”. A model of dry friction must contain a description of both phases. Different models have been proposed to describe dry friction; usually they differ only in the way the stick phase is modeled [10].

A realistic approach to control mechanical systems should be able to deal with the effects of dry friction [2, 20]. Dry friction can be described by either differential inclusions or by ordinary differential equations with discontinuous right-hand side [8, 25].

Systems with discontinuous elements exhibit a wide variety of complex phenomena which must be considered in the control design process [7, 12, 17]. For instance, these complex dynamical behaviors can generally result in vibration and instability that are highly undesirable in many cases [11]. Notwithstanding the impressive development of nonsmooth and set-valued analysis, these systems have not been closely studied either computationally or analytically [24].

For full-actuated mechanical systems, an effective approach to counteract the friction phenomenon has been the use of first-order sliding mode controllers. Discontinuous friction is

regarded as a bounded disturbance of unpredictable sign and therefore counteracted by choosing adequate control amplitude. These algorithms often require a high control effort to compensate this physical phenomenon and produce control signals that commute at theoretically infinite frequency, so it is not practical in many real situations [5]. Some techniques have been proposed to eliminate or attenuate this effect (e.g., sliding mode algorithms of higher order). However, the anti-chattering procedures, which aim at obtaining continuous control, do not necessarily guarantee accuracy in the presence of discontinuous friction [3, 4, 21].

The classical approach to control underactuated mechanical systems has commonly neglected the friction effect. Thus, in recent years several works have addressed the problem of friction in this class of systems. For example, linear damping (viscous friction) in the joints is considered in [1, 9, 26, 27]. When dry friction is present only in the controlled joint, the problem of compensation can be solved in some cases [15, 20, 22, 23].

However, the problem of controlling underactuated mechanical systems with dry friction in the non-actuated joint seems to be still open (see, e.g., [6, 13, 14, 20]). For instance, in [16, 17, 18] stick-slip oscillations and sticking phenomena of a class of underactuated mechanical systems are analyzed.

In this paper, we propose a dynamic continuous controller to regulate a class of full-actuated mechanical systems with dry friction.

The proposed controller makes use of the result presented in [13] concerning some stability conditions of mechanical systems with discontinuous friction, but it is robust with respect to more uncertainties and it operates in such a way that the objective of control is reached faster. Moreover, we propose a simpler method to find the parameters of the controller than the method presented in [13]. In addition, using this result, we propose a discontinuous controller for a class of 2-DOF underactuated mechanical systems with dry friction in the non-actuated joint. We illustrate this result with numerical examples of full-actuated systems and with an application to an experimental underactuated system.

## 2 Problem Statement

Consider a  $n$ -DOF mechanical system represented by

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= f_1(y_1, y_2) - C(y_2) \operatorname{sgn}(y_2) + u_c. \end{aligned} \quad (1)$$

Hereinafter,  $y_1 = (y_{11}, y_{12}, \dots, y_{1n})^T$  and  $y_2 = (y_{21}, y_{22}, \dots, y_{2n})^T$  are the generalized position vector and the velocity vector,  $f_1(y_1, y_2)$  is a smooth vector function,  $u_c \in \mathbb{R}^n$  is a control input vector, and  $\operatorname{sgn}(y_2)$  is the sign vector function, defined by

$$\operatorname{sgn}(y_2) = \begin{bmatrix} \operatorname{sgn}(y_{21}) \\ \operatorname{sgn}(y_{22}) \\ \vdots \\ \operatorname{sgn}(y_{2n}) \end{bmatrix}, \quad (2)$$

with

$$\operatorname{sgn}(y_{2i}) = \begin{cases} 1, & \text{for } y_{2i} > 0, \\ \epsilon, & \text{for } y_{2i} = 0, \\ -1, & \text{for } y_{2i} < 0, \end{cases} \quad i = 1, 2, \dots, n; \quad (3)$$

where  $\epsilon \in [-1, 1]$ , and  $C(y_2)$  is the matrix

$$C(y_2) = \begin{bmatrix} c_{11}(y_{21}) & 0 & \cdots & 0 \\ 0 & c_{22}(y_{22}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{nn}(y_{2n}) \end{bmatrix} \quad (4)$$

with the well-known friction model (see, e.g., [4, 19]) defined by

$$c_{ii}(y_{2i}) = c_{c_{ii}} + (c_{s_{ii}} - c_{c_{ii}}) \exp\left(-\frac{y_{2i}^2}{v_{s_{ii}}^2}\right), \quad i = 1, 2, \dots, n. \quad (5)$$

where  $c_{c_{ii}}$  and  $c_{s_{ii}}$  are the Coulomb friction level and the level of friction divided by a constant such that  $0 \leq c_{c_{ii}} \leq c_{s_{ii}}$ , and  $v_{s_{ii}}$  is the Stribeck velocity.

The control objective is to steer to zero both  $y_1(t)$  and  $y_2(t)$  by means of a continuous control vector  $u_c$ .

Note that (1) has  $n$  discontinuity surfaces  $\mathcal{S}_i$  characterized by  $y_{2_i} = 0$ ,  $i = 1, 2, \dots, n$ . Here, the meaning of such differential equation is viewed in the Filippov sense [8].

Let us consider a control law  $u_c$  with a dynamics given by

$$\dot{u}_c = \varphi(y_1, y_2, u_c) = \begin{bmatrix} \varphi_1(y_1, y_2, u_c) \\ \varphi_2(y_1, y_2, u_c) \\ \dots \\ \varphi_n(y_1, y_2, u_c) \end{bmatrix} \quad (6)$$

Note that  $u_c$  must be a continuous vector, but not necessarily  $\varphi$ . The system (1) (6) is then a system with the state  $(y_1, y_2, u_c)^T \in \mathbb{R}^{3n}$ . The next lemma will be a key result for what follows.

**Lemma 1.** For a given  $u_c \in \mathbb{R}^n$ , define  $\varphi_{0_i} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\varphi_{0_i}(y_1, u_c) = \varphi_i(y_1, 0, u_c)$  as the restriction of  $\varphi_i$  for  $y_2 = 0$ ,  $i = 1, 2, \dots, n$ . Suppose that  $y_{1_i} \neq 0$  implies that  $\varphi_{0_i}(y_1, u_c) \neq 0$ ,  $i = 1, 2, \dots, n$  for any  $u_c$ . Therefore, if (1) (6) exhibits the sliding mode in the intersection  $\mathcal{S}_0 = \mathcal{S}_1 \cap \mathcal{S}_2 \cap \dots \cap \mathcal{S}_n \setminus \{(0, 0, u_c)\}$ , then the system will leave this intersection in a finite time.

**Proof.** When the system (1) is in the sliding mode regime in the intersection  $\mathcal{S}_0 = \mathcal{S}_1 \cap \mathcal{S}_2 \cap \dots \cap \mathcal{S}_n \setminus \{(0, 0, u_c)\}$ , it is described by

$$y_2 = 0, \quad f_1(y_1, 0) - C(0)\text{sgn}(0) + u_c = 0. \quad (7)$$

Under this condition, at least  $y_{1_i}$  is a constant different from zero, hence  $\varphi_{0_i}(y_1, u_c) \neq 0$  while in the sliding mode. From (5),  $c_{ii}(0) = c_{s_{ii}}$ ; therefore, there exists a finite time in which the system leaves the sliding regime, that is,

$$u_{c_i} + f_{1_i}(y_1, 0) = \int_{t_0}^t \varphi_i(y_1, 0, u_c) d\tau + f_{1_i}(y_1, 0) \notin [-c_{s_{ii}}, c_{s_{ii}}] \quad (8)$$

where  $t_0$  is the initial time.

Now define  $x_1 = y_1$ ,  $x_2 = y_2$ , and  $x_3 = f_1(y_1, y_2) - C(y_2)\text{sgn}(y_2) + u_c$ , then the system (1) (6) is described, for  $x_{2_i} \neq 0$ ,  $i = 1, 2, \dots, n$ , by

$$\dot{x} = \begin{bmatrix} x_2 \\ x_3 \\ f_2(x_1, x_2, x_3) + F_4 x_3 + \dot{u}_c \end{bmatrix}, \quad (9)$$

$x = (x_1, x_2, x_3)^T$ ,  $f_2 = \dot{f}_1$ , and  $F_4$  is the matrix

$$F_4 = \begin{bmatrix} f_{4_{11}} & 0 & \dots & 0 \\ 0 & f_{4_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_{4_{nn}} \end{bmatrix} \quad (10)$$

with

$$f_{4_{ii}} = \frac{2(c_{s_{ii}} - c_{c_{ii}})}{v_{s_{ii}}^2} |x_{2_i}| \exp\left(-\frac{x_{2_i}^2}{v_{s_{ii}}^2}\right) \geq 0 \quad (11)$$

Since the control objective is to steer to zero both  $x_1(t)$  and  $x_2(t)$  by means of a continuous control  $u_c(t)$ , then the control aim is to regulate at zero the state  $x(t)$  of the system (9) with a hybrid "control"  $\dot{u}_c$ . This will be described in the next section.

### 3 Control Strategy

In this section, we present a control strategy that allows one to achieve the control objective.

Suppose that, for the system (9), there exists a control law

$$u_c = f_3(t, x_1, x_2), \quad (12)$$

such that  $\dot{u}_c \doteq \varphi(y, u_c)$  satisfies Lemma 1, where  $f_3(t, x_1, x_2)$  is a continuous vector function such that (9) with (12) can be transformed to

$$\dot{x} = \begin{bmatrix} x_2 \\ x_3 \\ -K_1 x_1 - K_2 x_2 - K_3 x_3 + F_4 x_3 + \dot{f}_5 + \tau_1 \end{bmatrix}, \quad (13)$$

where  $K_1$ ,  $K_2$ ,  $K_3$  are real-coefficient diagonal constant matrices, being  $K_1$  a positive diagonal matrix,

$$\dot{f}_5 = \begin{bmatrix} \dot{f}_{5_1}(x_{1_1}) \\ \dot{f}_{5_2}(x_{1_2}) \\ \vdots \\ \dot{f}_{5_n}(x_{1_n}) \end{bmatrix} \quad (14)$$

with

$$\frac{\partial f_{5_i}(x_{1_i})}{\partial x_{1_i}} < \infty, \quad i = 1, 2, \dots, n, \quad (15)$$

$$\tau_1 = \begin{bmatrix} \tau_{1_1} \\ \tau_{1_2} \\ \vdots \\ \tau_{1_n} \end{bmatrix} \quad (16)$$

with

$$\tau_{1_i} = \begin{cases} -k_{4_i} \operatorname{sgn}(x_{1_i}) & \text{if } x_{1_i} \neq 0 \text{ and } x_{2_i} = 0 \\ 0 & \text{if } x_{1_i} = 0 \text{ or } x_{2_i} \neq 0 \end{cases} \quad (17)$$

and  $k_{4_i} \geq 0$ ,  $i = 1, 2, \dots, n$ , are arbitrary real constants.

The next lemma shows that when the system is in the sliding mode with  $x_1 = 0$  then it remains there indefinitely, implying that  $\{0 \in \mathbb{R}^{3n}\}$  is an invariant set of the system (13).

**Lemma 2.** *Suppose that the system (1) (12) exhibits the sliding mode in the intersection of the surfaces  $S_i$ ,  $i = 1, 2, \dots, n$  and that  $x_1 = 0$ . Then the system will not leave this intersection.*

**Proof.** The system (1) (12) in the sliding mode, in the intersection of the surfaces  $S_i$ ,  $i = 1, 2, \dots, n$  with  $x_1 = 0$ , is given by

$$x_2 = 0, x_3 = -C(0)\operatorname{sgn}(0) + \int_{t_0}^t \tau_1 d\tau + f_5(0) + c_k = 0, \quad (18)$$

where  $t_0$  is the initial time and  $c_k$  is a finite constant vector. Since  $\tau_1 = 0$  for  $x_1 = 0$ , we have that

$$\int_{t_0}^t \tau_{1_i} d\tau + f_{5_i} + c_{k_i} \in [-c_{s_{ii}}, c_{s_{ii}}], \quad i = 1, 2, \dots, n, \quad (19)$$

for all time.

Therefore, if it is possible to find a function  $f_3$  such that the system (9) with (12) can be transformed to (13) and  $K_1, K_2, K_3$  forcing the state of (13) converge to zero, then the control objective will be attained.

Since  $\tau_{1_i} = 0$  for  $x_{2_i} \neq 0$ ,  $i = 1, 2, \dots, n$ , and it is possible to decouple this system into  $n$  1-DOF systems, the conditions for  $k_{1_{ii}}, k_{2_{ii}}, k_{3_{ii}}$ ,  $i = 1, 2, \dots, n$  to accomplish the objective  $\lim_{t \rightarrow \infty} \|x(t)\| = 0$  were presented in [13] and can be applied to this problem. This is summarized as follows.

Consider the matrices  $P_i$  and  $Q_i$  given by

$$P_i = \begin{bmatrix} p_{i11} & p_{i12} & p_{i13} \\ p_{i12} & p_{i22} & p_{i23} \\ p_{i13} & p_{i23} & p_{i33} \end{bmatrix}, \quad Q_i(x_{1_i}, x_{2_i}) = \begin{bmatrix} q_{i11} & q_{i12} \\ q_{i12} & q_{i22} \end{bmatrix}, \quad (20)$$

where the entries of  $Q_i(x_{1_i}, x_{2_i})$  are given by

$$q_{i11} = -2p_{i12} + 2p_{i23} \left[ k_{2_{ii}} - \frac{\partial f_{5_i}(x_{1_i})}{\partial x_{1_i}} \right], \quad (21)$$

$$q_{i12} = -p_{i22} + p_{i23} [k_{3_{ii}} - f_{4_{ii}}] + p_{i33} \left[ k_{2_{ii}} - \frac{\partial f_{5_i}(x_{1_i})}{\partial x_{1_i}} \right], \quad (22)$$

$$q_{i22} = -2p_{i23} + 2p_{i33} [k_{3_{ii}} - f_{4_{ii}}], \quad i = 1, 2, \dots, n. \quad (23)$$

Then the next theorem, shown in [13], can be applied.

**Theorem 3.** *Suppose that matrix  $Q_i$  is positive definite for all  $(x_{1_i}, x_{2_i})^T \in \mathbb{R}^2$ , matrix  $P_i$  is also positive definite and satisfies*

$$p_{i13} = 0, \quad \frac{p_{i11}}{p_{i23}} = \frac{p_{i12}}{p_{i33}} = k_{1_{ii}}, \quad (24)$$

for  $i = 1, 2, \dots, n$ . Then  $x = 0$  is a globally, asymptotically stable equilibrium point of the system (13).

In the next section, an application of this result to two systems is described.

## 4 Control Synthesis

In order to find the values of  $k_{1ii}$ ,  $k_{2ii}$ ,  $k_{3ii}$ ,  $i = 1, 2, \dots, n$  easily, we propose  $P_i$  and  $Q_i(x_{1i}, x_{2i})$  given by

$$P_i = \frac{1}{2} \begin{bmatrix} \alpha_i k_{1ii} & k_{1ii} & 0 \\ k_{1ii} & \alpha_i k_{3ii} + k_{2ii} & \alpha_i \\ 0 & \alpha_i & 1 \end{bmatrix}, \quad (25)$$

$$Q_i(x_{1i}, x_{2i}) = \begin{bmatrix} q_{i11} & q_{i12} \\ q_{i12} & q_{i22} \end{bmatrix}$$

where

$$q_{i11} = -k_{1ii} + \alpha_i \left[ k_{2ii} - \frac{\partial f_{5i}(x_{1i})}{\partial x_{1i}} \right], \quad (26)$$

$$q_{i12} = -\frac{1}{2} \alpha_i f_{4ii} - \frac{1}{2} \frac{\partial f_{5i}(x_{1i})}{\partial x_{1i}}, \quad (27)$$

$$q_{i22} = -\alpha_i + k_{3ii} - f_{4ii}. \quad (28)$$

Therefore, Theorem 1 is satisfied if

$$\alpha_i > 0 \quad (29)$$

$$k_{1ii} < \alpha_i^2 k_{3ii} + \alpha_i k_{2ii} - \alpha_i^3 \quad (30)$$

$$k_{2ii} > \frac{k_{1ii}}{\alpha_i} + \left| \frac{\partial f_{5i}(x_{1i})}{\partial x_{1i}} \right| \quad (31)$$

$$k_{3ii} > \left( \frac{1}{2} \alpha_i f_{4ii} + \frac{1}{2} \left| \frac{\partial f_{5i}(x_{1i})}{\partial x_{1i}} \right| \right)^2 / (\alpha_i [k_{2ii} - \left| \frac{\partial f_{5i}(x_{1i})}{\partial x_{1i}} \right|] - k_{1ii}) + \alpha_i + f_{4ii} \quad (32)$$

where

$$f_{4ii} < \frac{2(c_{s_{ii}} - c_{c_{ii}})}{v_{s_{ii}}} (0.429), \quad i = 1, 2, \dots, n. \quad (33)$$

concluding that  $x = 0$  is a globally, asymptotically stable equilibrium point of system (13).

Note that in order to find the parameters  $k_{1ii}$ ,  $k_{2ii}$ ,  $k_{3ii}$ ,  $i = 1, 2, \dots, n$ , it is not required to know  $c_{s_{ii}}$ ,  $c_{c_{ii}}$ ,  $v_{s_{ii}}$ ,  $\frac{\partial f_{5i}(x_{1i})}{\partial x_{1i}}$  exactly.

In what follows, we describe the controller design procedure using two examples to regulate the position of a mechanical system. The first example is a pendulum with dry friction; the second example is an experimental torsional system with dry friction.

### Example 1. A Pendulum

Let us consider the system shown in Figure 1, described by

$$ml^2 \ddot{q} + mgl \sin(q) + ml^2 c(\dot{q}) \operatorname{sgn}(\dot{q}) = u_1, \quad (34)$$

where  $q \in \mathbb{R}$  is the angular position,  $\dot{q} \in \mathbb{R}$  the angular velocity,  $m$  the mass,  $l$  is the distance,  $c$  is given by (4), and  $u_1 \in \mathbb{R}$ .

The objective is to design a continuous control law  $u_1$  so that the position  $q$  converges to a given constant value  $q_d$ .

For this system we have  $f_5(x_1) = f_1(x_1, x_2) = -\frac{g}{l} \sin(x_1 + q_d)$  and  $u_c = \frac{1}{ml^2} u_1$  (see Eq. (1)), where  $x_1 = q - q_d$ ,  $x_2 = \dot{q}$ . If

$$u_c = - \int_{t_0}^t [k_1 x_1(\tau) - \tau_1(\tau)] d\tau - k_2 x_1 - k_3 x_2, \quad (35)$$

where  $t_0$  is the initial time and  $\tau_1$  is given by (16), then we arrive at the desired form (13).

Finally, if we find the constants  $k_i$ ,  $i = 1, 2, 3$ , which satisfy Theorem 3, then the control objective will be attained.

Figure 2 shows numerical results, where we have set  $m = 1$ ,  $l = 1$ ,  $g = 9.81$ ,  $c_s = 1$ ,  $c_c = 0.75$ ,  $v_s = 0.143$ . We propose  $\alpha = 1$ ,  $k_1 = 0.5$  and, from (29), (30), (31), (32), and (33),  $k_2 = 11$ ,  $k_3 = 47$ ,  $k_4 = 20$ , with  $q(t_0) = 0$ ,  $\dot{q}(t_0) = 0$  and  $q_d = \pi$ .

### Example 2. An Experimental Torsional System

Let us consider the experimental system configuration model 205 of Educational Control Products (ECP) shown in Figure 3, described by

$$\begin{aligned} J_1 \ddot{q}_1 + k_a (q_1 - q_2) + f_a \dot{q}_1 &= u_1, \\ J_2 \ddot{q}_2 - k_a (q_1 - q_2) + f_b \dot{q}_2 + J_2 c (\dot{q}_2) \operatorname{sgn}(\dot{q}_2) &= 0, \end{aligned} \quad (36)$$

where  $q_i \in \mathbb{R}$  are the angular positions,  $\dot{q}_i \in \mathbb{R}$  the angular velocities,  $J_i$  the inertial, for  $i = 1, 2$ ;  $c$  is given by (4),  $k_a$ ,  $f_a$  and  $f_b$  are viscous friction coefficients, and  $u_1 \in \mathbb{R}$ .

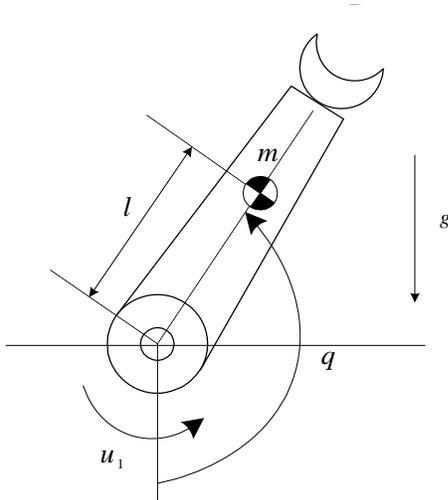


Fig. 1. A pendulum

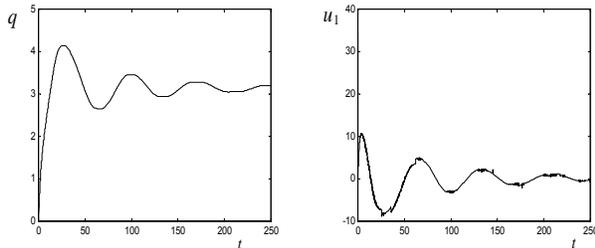


Fig. 2. Position and control of the pendulum

This system consists of two plates, without anchoring, coupled by a torsional spring. Control input is at plate 1, and dry friction in joint 2 is introduced through a DC motor attached to plate 2.

The objective is to design a control law  $u_1$  so that the angle  $q_2$  converges to a given constant value  $q_{2d}$ , with  $q_1$  and  $\dot{q}_1$  bounded.

If we define  $x_1 = q_2 - q_{2d}$ , where  $q_{2d}$  is a desired constant position,  $x_2 = \dot{q}_2$ ,  $x_3 = \ddot{q}_2$ ,

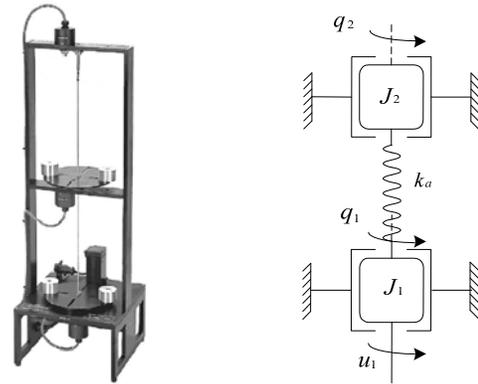


Fig. 3. Torsional system

$$u_c = \frac{k_a}{J_2} q_1, \quad (37)$$

and

$$f_1(x_1, x_2) = -\frac{k_a}{J_2} q_2 - \frac{f_b}{J_2} \dot{q}_2, \quad (38)$$

then the unactuated joint is described by the equations

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f_1(x_1, x_2) - c(x_2) \operatorname{sgn}(x_2) + u_c = x_3, \end{aligned} \quad (39)$$

which have the form of the system (1), so Theorem 3 can be applied to design an ideal expression for the virtual control  $\dot{u}_c$ , equation (37), which we denote by  $\dot{u}_c^*$ . This is given by

$$\dot{u}_c^* = -k_1 x_1 - k_{c2} x_2, \quad (40)$$

with  $k_{c2} = k_2 - k_a/J_2$  and  $\tau_1 = 0$ . Here,  $k_3 = f_b/J_2$ . Note that if the state  $x = (x_1, x_2, x_3)^T$  is bounded then  $u_c$  will also be bounded and, from (37) and (40), the position and speed of the actuated joint remain bounded.

Now the control  $u_1$  can be designed using the classic technique of sliding modes. This is convenient because if we use  $s = \dot{u}_c - \dot{u}_c^*$  as a

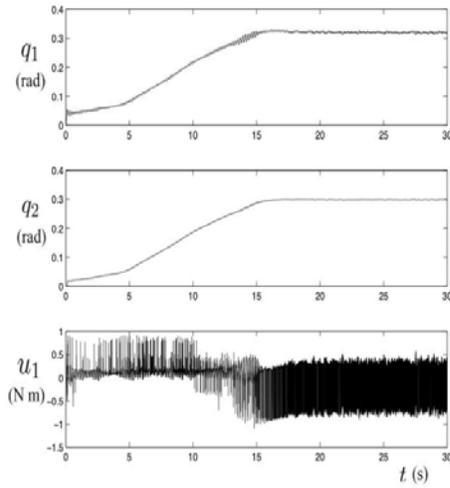


Fig. 4. Position and control of the torsional system

sliding surface then a sliding mode control designed for  $u_1$  makes the control  $\dot{u}_c$  converge to  $\dot{u}_c^*$  in finite time. Once  $\dot{u}_c = \dot{u}_c^*$ , Theorem 3 ensures the convergence of  $x_1 = q_2 - q_{2d}$  to zero.

So let us propose a control law  $u_1$  be given by

$$u_1 = k_a (q_1 - q_2) + f_a \dot{q}_1 + J_1 \tau_2, \quad (41)$$

then the actuated joint of system (36) takes the form

$$\ddot{q}_1 = \tau_2, \quad (42)$$

and let the sliding surface be defined by

$$\begin{aligned} s &= \dot{u}_c - \dot{u}_c^* \\ &= \frac{k_a}{J_2} \dot{q}_1 + k_1 x_1 + k_{c2} x_2, \end{aligned} \quad (43)$$

rendering the sliding mode controller given by (41) with

$$\tau_2 = \frac{J_2}{k_a} \left[ -k_5 \text{sgn}(s) - k_1 \dot{q}_2 - k_{c2} \left( -\frac{k_a}{J_2} q_2 - \frac{f_b}{J_2} \dot{q}_2 + \frac{k_a}{J_2} q_1 \right) \right] \quad (44)$$

with  $k_5 > |k_{c2}| c_s$ .

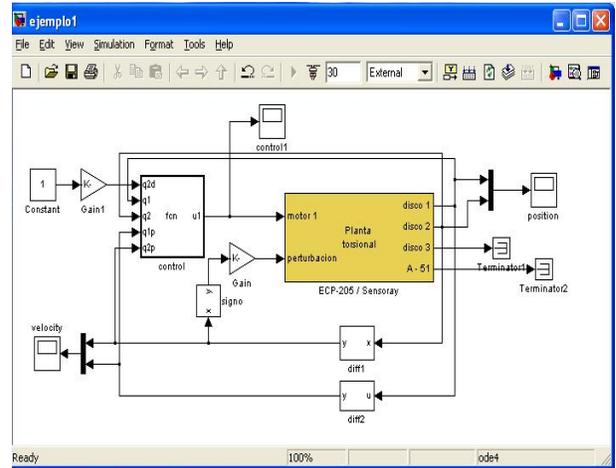
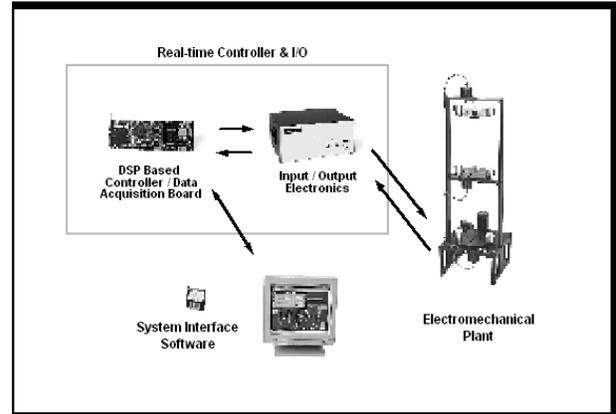


Fig. 5. Interface of the system

Finally, if we find the constants  $k_i$ ,  $i = 1, 2, 3$ , which satisfy Theorem 3, then the control objective will be attained.

Figure 4 shows the experimental results. According to the manual and the identification of system parameters, the inertia are  $J_1 = 0.0193$  Nms<sup>2</sup>/rad and  $J_2 = 0.0187$  Nms<sup>2</sup>/rad, the coefficient of elasticity of the spring is  $k_a = 3.2178$  Nm/rad, and the coefficients of viscous friction are  $f_a = 0.1373$  Nms/rad and  $f_b = 0.3$  Nms/rad. Note that  $k_3 = f_b/J_2$ . We propose  $c_s = c_c = 3.7$  rad/s<sup>2</sup>,  $v_s = 1$  rad/s,  $\alpha = 4$ ,  $k_1 = 4500$ , and from (29), (30), (31), (32), and (33)  $k_2 = 1500$ ,  $k_5 = 5000$ ,

$q_1(t_0) = 0$  rad,  $\dot{q}_1(t_0) = 0$  rad/s,  $q_2(t_0) = 0$  rad,  $\dot{q}_2(t_0) = 0$  rad/s, and  $q_{2d} = 0.3$  rad.

Figure 5 shows the user interface of the torsional system using Matlab Simulink.

## 5 Conclusions

In this paper, we have proposed a continuous controller for a class of full-actuated mechanical systems with dry friction. The proposed controller makes use of the result presented in [13], but it is robust with respect to uncertainties in the parameter values of the system and it operates in the way that the objective of control is reached faster, since the system leaves the obstruction faster. Moreover, we have proposed a simpler method to find the parameters of the controller than the method presented in [13].

In addition, using this result, we have shown its application to control of an underactuated mechanical system with dry friction in the non-actuated joint. In this case, the control objective is to regulate the non-actuated joint while the position and speed of the actuated joint remain bounded. Since the term in the non-actuated joint containing dry friction must be compensated by a continuous action, a discontinuous control is designed using the classic technique of sliding modes. The proposed controller guarantees the convergence of the position error of the non-actuated joint to zero. We illustrated these results with an application to control an experimental torsional system.

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**Roque Martínez** received Engineering degree in Communication and Electronic Engineering from Zacatecas Autonomous University (UAZ), Mexico, in 1997, and M.Sc. and Sc.D. degrees in Electronics and Telecommunications with orientation in Instrumentation and Control from Scientific Research and Advanced Studies Center of Ensenada (CICESE), Mexico, in 2002 and 2007, respectively. He has been a full-time professor of Mechanic Engineering Program at the Engineering Department of UAZ since 2008. His research interests are in the fields of nonlinear control and discontinuous mechanical systems.



**Joaquín Álvarez** received Engineering degree in Electronics and Telecommunication from the School of Electrical and Mechanical Engineering of the National Polytechnic Institute (IPN) of Mexico in 1975. He is MSc by CINVESTAV-IPN, Mexico (1976) and Doctor in Engineering (PhD) in Automatic Control by the National School of Electrotechnical Engineering of the National Polytechnic Institute of Grenoble, France (1979). He held a professorial position at the Electrical Engineering Department of the Advanced Studies and Research Center of IPN. Since 1990, he has been working as a full-time professor at the Electronics and Telecommunications Department of Scientific Research and Advanced Studies Center of Ensenada (CICESE), Mexico, where he has been appointed as the Head of this Department. At present, his research interests are in the fields of nonlinear control, discontinuous systems, chaos control and synchronization, with applications to mechanical systems and electronic circuits.

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