A New LU Decomposition on Hybrid GPU-Accelerated Multicore Systems

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Abstract. In this paper, we postulate a new decomposition theorem of a matrix **A** into two matrices, namely, a lower triangular matrix **M**, in which all entries are determinants, and an upper triangular matrix **U** whose entries are also in determinant form. From a well-known theorem on the pivot elements of the Doolittle-Gauss elimination process, we deduce a corollary to obtain a diagonal matrix **D**. With it, we scale the elementary lower triangular matrix of the Doolittle-Gauss elimination process and deduce a new elementary lower triangular matrix. Applying this linear transformation to **A** by means of both minimum and complete pivoting strategies, we obtain the determinant of **A** as if it had been calculated by means of a Laplace expansion. If we apply this new linear transformation and the above pivot strategy to an augmented matrix (**A|b**), we obtain a Cramer's solution of the linear system of equations. These algorithms present an $O(n^3)$ computational complexity when $(\mathbf{A,b}) \subset \mathbf{R^n}$ on hybrid GPU-accelerated multicore systems.

Keywords. New LU theorem, Cramer rule, Gauss elimination, Laplace expansion, determinants, GPU, multicore systems.

Una nueva descomposición LU calculada en sistemas multi-core acelerados con GPU

Resumen. En este trabajo se postula un Nuevo Teorema de Descomposición de una Matriz A en dos matrices: una Matriz triangular inferior M, cuyas entradas son todas expresadas en forma de determinantes, y una matriz triangular superior U cuyas entradas están también expresadas en forma de determinantes. A partir de un muy conocido Teorema sobre los elementos pivotales del proceso de eliminación de Doolittle-Gauss, deducimos un corolario para obtener una Matriz Diagonal D. Usando esta matriz, escalamos la Matriz Elemental Triangular Inferior obtenida durante el proceso de eliminación de Doolittle-Gauss y deducimos una Nueva Matriz Elemental Triangular Inferior. Aplicando esta transformación lineal a la matriz A, por medio de una estrategia de pivoteo total, se obtiene el determinante de A como si hubiera sido calculado a través de la Expansión de Laplace. Si aplicamos esta nueva transformación lineal y la estrategia de pivoteo anteriormente mencionada a la matriz aumentada (A|b) obtenemos la solución de la Regla de Cramer aplicada a un Sistema de Ecuaciones Lineales. Estos algoritmos presentan una complejidad computacional $O(n^3)$ cuando $(\mathbf{A,b}) \subset \mathbf{R^n}$ se calcula en Sistemas Multi-Core Acelerados con GPU.

Palabras clave. Nuevo teorema LU, regla de Cramer, eliminación de Gauss, expansión de Laplace, determinantes, GPU, sistemas Multi-Core.

1 Introduction

A Linear System of Equations (LSE) can be defined as a set of *m* equations with *n* unknowns represented by a matrix **A**, a vector **b** and an unknown vector **x**, namely, $\mathbf{Ax} = \mathbf{b}$. Many methods have been proposed to solve such linear equations. A famous one is Cramer's rule, where each component of the solution is determined as the ratio of two determinants.

When trying to solve a system of *n* equations using Cramer's rule, one needs to compute *n+1* determinants, each of order *n*. If these are computed in a straightforward way, using the Laplace expansion, the solution to the linear system takes $(n+1)(n!)(n-1)$ multiplications, plus a similar number of additions. Although Cramer´s rule possesses a fundamental theoretical importance, it may result impractical in computations. It is for that reason that this method is seldom recommended [1, 6]. Cramer´s rule has at least one attractive property: it computes every element of the solutions independently. For this reason, it can be a practical method for some special linear systems on parallel computers [7].

There are much better ways to compute determinants. In this paper, we propose a new method that renders it possible to solve a linear system in about the same time as it takes to compute one determinant.

Another approach, with a certain mathematical appeal but considerable computational pitfalls, finds the solution to a linear system of equations using the inverse matrix $\mathbf{A}^{-1}$. However, in virtually every application, it is unnecessary and inadvisable to compute the inverse matrix explicitly. The inverse matrix method requires more arithmetic and produces a less accurate answer. Therefore, neither of the above methods is recommended [8].

The Gaussian Transformation (GT) for solving an LSE has proved to be the best option for most practical applications. The new transformation proposed here can be obtained from it. In the next section we briefly review this topic.

2 LU-Matricial Decomposition with GT Notation and Definitions

The problem of solving a linear system of equations $\mathbf{Ax}=\mathbf{b}$ is central to the field of matrix computation. There are several ways to perform the elimination process necessary for its matrix triangulation. We will focus on the Doolittle-Gauss elimination method: application of the algorithm of choice when **A** is square, dense, and unstructured.

Let us assume that $\mathbf{A}\in\mathbf{R}^{\mathbf{nxn}}$ is nonsingular and that we wish to solve the linear system $\mathbf{Ax}=\mathbf{b}$. Here we show how for exact arithmetic, partial pivoting and column interchanges, some Gauss transformations $\mathbf{M_1},...,\mathbf{M_{n-1}}$ can almost always be found such that $\mathbf{M_{n-1}},...,\mathbf{M_2M_1A}=\mathbf{U}$ is upper triangular [9]. The original $\mathbf{Ax}=\mathbf{b}$ problem is then equivalent to the upper triangular system $\mathbf{Ux}=(\mathbf{M_{n-1}},...,\mathbf{M_2M_1})\ \mathbf{b}$ which can be solved through back-substitution.

Suppose then, that $\mathbf{A}\in\mathbf{R}^{\mathbf{nxn}}$ and that for some *k<n* we have determined the Gauss transformations $\mathbf{M_1},...,\mathbf{M_{k-1}}\in\mathbf{R}^{\mathbf{nxn}}$ such that

$$\mathbf{A}^{(\mathbf{k-1})}\equiv\mathbf{M}_{\mathbf{k-1}}\cdots\mathbf{M}_1\mathbf{A}=\begin{array}{cc} \begin{pmatrix}\mathbf{A}_{11}{}^{(\mathbf{k-1})} & \mathbf{A}_{12}{}^{(\mathbf{k-1})}\\ \mathbf{0} & \mathbf{A}_{22}{}^{(\mathbf{k-1})}\end{pmatrix} & \begin{array}{l}(k-1)\\(n-k+1)\end{array}\\ \begin{array}{cc}(k-1) & (n-k+1)\end{array} & \end{array} \quad (1)$$

where $\mathbf{A_{11}}^{(\mathbf{k-1})}$ is an upper triangular matrix.

Now, if

$$\mathbf{A}_{22}{}^{(\mathbf{k-1})}=\begin{pmatrix} a_{kk}{}^{(k-1)} & . & . & . & a_{kn}{}^{(k-1)}\\ . & & & & .\\ . & & & & .\\ . & & & & .\\ a_{nk}{}^{(k-1)} & . & . & . & a_{nn}{}^{(k-1)}\end{pmatrix} \quad (2)$$

and $a_{kk}{}^{(k-1)}\neq 0$,

then the multiplicators

$$m_i=\frac{a_{ik}{}^{(k-1)}}{a_{kk}{}^{(k-1)}};i=k+1,\ldots,n\ ,$$

with $a_{kk}\neq 0$, are well defined.

So, we obtain what follows.

Definition. An elementary lower triangular matrix of order *n* and index *k* is a matrix of the form [10]

$$\mathbf{M_k}=\mathbf{I_n}-\mathbf{me_k^T} \quad (3)$$

where

$$\mathbf{e_k^T} = (0,\ldots,0,\underset{k}{1},0,\ldots,0)^T, \quad \mathbf{I_n} = \begin{pmatrix} 1 & . & . & . & 0 \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ 0 & . & . & . & 1 \end{pmatrix} \tag{4}$$

$$\mathbf{m^T} = (\underbrace{0,0,0,\ldots,0}_{k-times}, m_{k+1},\ldots,m_n)$$

In general, an elementary lower triangular matrix has the above form.

The computational significance of elementary lower triangular matrices is that they can be used to introduce zero components into a vector. Thus,

$$\mathbf{M_k} \cdot \begin{pmatrix} a_{11} \\ . \\ . \\ . \\ a_{k1} \\ a_{k+1,1} \\ . \\ . \\ . \\ a_{n1} \end{pmatrix} = \begin{pmatrix} a_{11} \\ . \\ . \\ . \\ a_{k1} \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \tag{5}$$

The matrix $\mathbf{M_k}$ is said to be a GT**.** The vector **m** is referred to as the Gauss vector. The components of **m** are known as multipliers.

Then it follows that

$$\mathbf{A^{(k)}} \equiv \mathbf{M_k A^{(k-1)}} = \begin{array}{c} \begin{pmatrix} \mathbf{A_{11}}^{\mathbf{(k)}} & \mathbf{A_{12}}^{\mathbf{(k)}} \\ \mathbf{0} & \mathbf{A_{22}}^{\mathbf{(k)}} \end{pmatrix} \begin{matrix} (k) \\ (n-k) \end{matrix} \\ \begin{matrix} (k) & (n-k) \end{matrix} \end{array} \tag{6}$$

where $\mathbf{A_{11}}^{\mathbf{(k)}}$ is an upper triangular matrix.

This process illustrates the *k*-th step of the decomposition process, in which we used

$$(\mathbf{M_k} \cdots \mathbf{M_1})^{-1} = \mathbf{M_1}^{-1} \cdots \mathbf{M_k}^{-1} = \prod_{i=1}^{k} (\mathbf{I_n} + \mathbf{m^{(i)}} \mathbf{e_i}^{\mathbf{T}}) = \mathbf{I_n} + \sum_{i=1}^{k} \mathbf{m^{(i)}} \mathbf{e_i^T} \tag{7}$$

We find the final expression for the decomposition process as

$$\begin{gathered} \mathbf{A} = \begin{pmatrix} \mathbf{L_{11}}^{\mathbf{(k)}} & \mathbf{0} \\ \mathbf{L_{21}}^{\mathbf{(k)}} & \mathbf{I_{n-k}} \end{pmatrix} \begin{pmatrix} \mathbf{I_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{A_{22}}^{\mathbf{(k)}} \end{pmatrix} \begin{pmatrix} \mathbf{A_{11}}^{\mathbf{(k)}} & \mathbf{A_{12}}^{\mathbf{(k)}} \\ \mathbf{0} & \mathbf{I_{n-k}} \end{pmatrix} \Rightarrow \\ (\mathbf{M_k \ldots M_1})^{-1} \equiv \begin{pmatrix} \mathbf{L_{11}}^{\mathbf{(k)}} & \mathbf{0} \\ \mathbf{L_{21}}^{\mathbf{(k)}} & \mathbf{I_{n-k}} \end{pmatrix} = \mathbf{I_n} + (\mathbf{m^{(1)}},\ldots,\mathbf{m^{(k)}},0,\ldots,0). \end{gathered} \tag{8}$$

In general, forward elimination consists of *n-1* steps. At the *k-th* step, multiples of the *k-th* equation are subtracted from the remaining equations to eliminate the *k-th* variable. If the pivot element $a_{k,k}^{(k)}$ is null or “small”, it is advisable to interchange equations before this is done through **P**, a permutation matrix that records the row exchanges as detailed below.

2.1 LU Decomposition Theorem

Using the above expression, the following can be established [11]:

Theorem. Let $\mathbf{A_k}$ denote the leader or main sub-matrix (*k* x *k*) of $\mathbf{A} \in \mathbf{R^{nxn}}$. If $\mathbf{A_k}$ is non-singular for *k=1,...,n* , then there exists a lower triangular matrix $\mathbf{L} \in \mathbf{R^{nxn}}$ and an upper triangular matrix $\mathbf{U} \in \mathbf{R^{nxn}}$ so that **A=PLU.** Furthermore, $\left|\mathbf{A_k}\right| = u_{11} \ldots u_{kk} \forall k = 1,\ldots,$ *n.*

3 Derivation of a New Linear Transformation

In the previous section we have defined an elementary lower triangular matrix of order *n* and index *k* as $\mathbf{M_k} = \mathbf{I_n} - \mathbf{me_k^T}$. We shall find that the above expression and the next theorem**,** whose proof can be found in [10, 11], are useful.

Theorem. The pivot elements $a_{k,k}^{(k)} \quad (k = 1,\ldots, n)$ are nonzero if and only if the leading principal submatrices $\mathbf{A_k} \quad (k = 1,\ldots, n)$ are non-singular.

On this basis, the following can be stated.

Corollary. Let $\mathbf{A} \in \mathbf{R}^{\mathbf{nxn}}$ be a matrix with $\mathbf{A_k}$ $(k = 1,\dots,n)$ non-singular leading principal sub-matrices. Then, there exists a unique diagonal matrix for *k=1* $(a_{00}^{(1)} = 1)$ whose entries are

$$\mathbf{D_k} = \begin{pmatrix} 1_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & 1_k & & & & \\ & & & & \left(\frac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_{k+1} & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & \left(\frac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_n \end{pmatrix} \quad (9)$$

where $a_{k,k}^{(k)}$ and $a_{k-1,k-1}^{(k)}$ are the pivot elements.

Now, if we scale the matrix $\mathbf{M_k}$ with that diagonal matrix, we obtain

$$\mathbf{M'_k} = \mathbf{D_k M_k} =$$

$$= \begin{pmatrix} 1_1 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 1_k & & & \\ & & & -\frac{a_{k+1,k}^{(k)}}{a_{k-1,k-1}^{(k)}} & \left(\frac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_{k+1} & & \\ & & & \cdot & & \cdot & \\ & & & \cdot & & & \cdot \\ & & & -\frac{a_{n,k}^{(k)}}{a_{k-1,k-1}^{(k)}} & \cdot & \cdot\ \cdot & \left(\frac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_n \end{pmatrix} \quad (10)$$

and, once simplified, it can be re-expressed as presented in (11).

By using this new transformation and applying the elimination process to a matrix **A**, all of whose entries are integers, all intermediate results are integers too, forming a number ring [12], since they are obtained through additions and products. Thus, $\mathbf{A} \in \mathbf{I}^{\mathbf{nxn}} \Rightarrow (\mathbf{M'}, \mathbf{U}) \in \mathbf{I}^{\mathbf{nxn}}$.

$$\mathbf{M'_k} = \frac{1}{a_{k-1,k-1}^{(k)}} \cdot \begin{pmatrix} \left(a_{k-1,k-1}^{(k)}\right)_1 & & & & & \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \left(a_{k-1,k-1}^{(k)}\right)_k & & \\ & & & -a_{k+1,k}^{(k)} & \left(a_{k,k}^{(k)}\right)_{k+1} & \\ & & & \cdot & \cdot & \\ & & & \cdot & & \cdot \\ & & & -a_{n,k}^{(k)} & \cdots & \left(a_{k,k}^{(k)}\right)_n \end{pmatrix} \quad (11)$$

The products of these results times the factor $\frac{1}{a_{k-1,k-1}^{(k)}}$ are integers too, because previously, in the $k-1$ step of the elimination process, such results had been multiplied by $a_{k-1,k-1}^{(k)}$ .This multiplication process leads to a simplification of the final result.

We can re-express this linear transformation as $(a_{00}^{(1)} = 1)$.

3.1 New Decomposition Theorem through Determinants

Using any of the above expressions, one can state the following theorem.

Theorem. Let $\mathbf{A} \in \mathbf{R}^{\mathbf{nxn}}$ and $\mathbf{M} = \prod_{\mathbf{k=n-1}}^{\mathbf{1}} \mathbf{M'_k}$ .Then **M** is a lower triangular matrix all of whose components are determinants, and $\mathbf{U} = \mathbf{MA}$ also has all components in determinant form.

$$\mathbf{M'k} = \begin{pmatrix} \frac{1}{a_{k,k}^{(k)}}\mathbf{Ik} & 0 \\ 0 & \frac{1}{a_{k-1,k-1}^{(k)}}\mathbf{In-k} \end{pmatrix} \left[a_{k,k}^{(k)}\mathbf{I} - \begin{pmatrix} 01 \\ \cdot \\ \cdot \\ 0k \\ a_{k+1,k}^{(k)} \\ \cdot \\ \cdot \\ a_{n,k}^{(k)} \end{pmatrix} \mathbf{e_k^T} \right] ; \forall k = 1,\dots,n-1 \quad (12)$$

Furthermore, $|\mathbf{A}| = u_{n,n}$

Proof. It follows from induction on *n*. For *n=1* the theorem is trivially true since $\mathbf{M} = \mathbf{M}_1^{'} = 1$ and $\mathbf{U} = \mathbf{MA} = a_{11}$. For the induction step $(k = 1,..., n-1)$ we have

n=2
k=1

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix};\ \mathbf{M}_1^{'} = \begin{pmatrix} 1 & 0 \\ -a_{21} & a_{11} \end{pmatrix};$$
$$\mathbf{M} = \mathbf{M}_1^{'};$$
$$\mathbf{U} = \mathbf{MA} = \begin{pmatrix} 1 & 0 \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix} \tag{13}$$

n=3
k=1,2

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};$$
$$\mathbf{M}_1^{'} = \begin{pmatrix} 1 & 0 & 0 \\ -a_{21} & a_{11} & 0 \\ -a_{31} & 0 & a_{11} \end{pmatrix};$$
$$\mathbf{U}_1 = \mathbf{M}_1^{'}\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{pmatrix} \tag{14}$$

$$\mathbf{M}_2^{'} = \frac{1}{a_{11}} \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix} \tag{15}$$

$$\mathbf{M} = \mathbf{M}_2^{'}\mathbf{M}_1^{'} = \begin{pmatrix} 1 & 0 & 0 \\ -a_{21} & a_{11} & 0 \\ -\begin{vmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix} \tag{16}$$

The last row of the matrix **M** multiplied by the last column of the matrix **A** is equivalent to Laplace expansion of **A** taking out the last column. Then we have $u_{33} = |\mathbf{A}|$.

$$\mathbf{U} = \mathbf{MA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ 0 & 0 & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{pmatrix} \tag{17}$$

n
k=1,2,…,n-1

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n-1} & a_{1,n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{1,n-1} & a_{2,n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3,n-1} & a_{3,n} \\ . & . & . & . & & & . & . \\ . & . & . & & . & & . & . \\ . & . & . & & & . & . & . \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & . & . & . & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & . & . & . & a_{n,n-1} & a_{n,n} \end{pmatrix} \tag{18}$$

$$\mathbf{M} = \mathbf{M}_{n-1}^{'} \ldots \mathbf{M}_1^{'} =$$
$$= \begin{pmatrix} m_{11} & 0 & 0 & . & . & . & 0 & 0 \\ m_{21} & m_{22} & 0 & . & . & . & 0 & 0 \\ m_{31} & m_{32} & m_{33} & . & . & . & 0 & 0 \\ . & . & . & . & & & & \\ . & . & . & & . & & & \\ . & . & . & & & . & & \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & . & . & . & m_{n-1,n-1} & 0 \\ m_{n,1} & m_{n,2} & m_{n,3} & . & . & . & m_{n,n-1} & m_{n,n} \end{pmatrix}$$

where

$$m_{11} = 1;\; m_{21} = -a_{21}\;;$$

$$m_{22} = a_{11};\; m_{31} = -\begin{vmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \end{vmatrix};$$

$$m_{32} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix};\; m_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix};$$

$$m_{n-1,1} = -\begin{vmatrix} a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} \end{vmatrix}$$

$$m_{n-1,2} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} \end{vmatrix} \qquad (19)$$

$$m_{n-1,3} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} \end{vmatrix}$$

$$m_{n-1,n-1} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} \end{vmatrix}$$

$$m_{n,1} = -\begin{vmatrix} a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix}$$

$$m_{n,2} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix}$$

$$m_{n,3} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix}$$

$$m_{n,n-1} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} \end{vmatrix}$$

$$m_{n,n} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix}$$

and

$$\mathbf{U}=\mathbf{MA}=\begin{pmatrix} u_{11} & u_{12} & u_{13} & . & . & . & u_{1,n-1} & u_{1,n} \\ 0 & u_{22} & u_{23} & . & . & . & u_{2,n-1} & u_{2,n} \\ 0 & 0 & u_{33} & . & . & . & u_{3,n-1} & u_{3,n} \\ . & . & . & . & & & . & . \\ . & . & . & & . & & . & . \\ . & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & . & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & 0 & . & . & . & 0 & u_{n,n} \end{pmatrix} \tag{20}$$

where

$$u_{11}=a_{11}\ ;\ u_{12}=a_{12}\ ;\ u_{13}=a_{13}\ ;$$
$$u_{1,n-1}=a_{1,n-1}\ ;\ u_{1,n}=a_{1,n}\ ;$$
$$u_{22}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix};u_{23}=\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix};$$
$$u_{2,n-1}=\begin{vmatrix} a_{11} & a_{1,n-1} \\ a_{21} & a_{2,n-1} \end{vmatrix};u_{2,n}=\begin{vmatrix} a_{11} & a_{1,n} \\ a_{21} & a_{2,n} \end{vmatrix}$$
$$u_{33}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{21}$$
$$u_{3,n-1}=\begin{vmatrix} a_{11} & a_{12} & a_{1,n-1} \\ a_{21} & a_{22} & a_{2,n-1} \\ a_{31} & a_{32} & a_{3,n-1} \end{vmatrix}$$
$$u_{3,n}=\begin{vmatrix} a_{11} & a_{12} & a_{1,n} \\ a_{21} & a_{22} & a_{2,n} \\ a_{31} & a_{32} & a_{3,n} \end{vmatrix}$$

The Laplace expansion of sub-matrix $\mathbf{A_{n-1,n-1}}$ taking out the last column is equivalent to multiplication of the *(n-1)-th* row of the matrix **M** by the *(n-1)-th* column of **A**. Then we have $u_{n-1,n-1}=\left|\mathbf{A_{n-1,n-1}}\right|$ and

$$u_{n-1,n-1}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3,n-1} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & . & . & . & a_{n-1,n-1} \end{vmatrix} \tag{22}$$

In a similar way, we have

$$u_{n-1,n}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2,n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3,n} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & . & . & . & a_{n-1,n} \end{vmatrix} \tag{23}$$

Finally, taking the last row of **M** multiplied by the last column of **A** we have

$$u_{n,n}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2,n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3,n} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n,1} & a_{n,2} & a_{n,3} & . & . & . & a_{n,n} \end{vmatrix} \tag{24}$$

Now, if **M** is the new transformation, then we obtain

$$\mathbf{M}=\prod_{\mathbf{k=n-1}}^{\mathbf{1}}\mathbf{M}_{\mathbf{k}}^{'}\ ;\ \mathbf{U}=\mathbf{MA} \tag{25}$$

In order to solve the linear system of equations $\mathbf{Ax}=\mathbf{b}$, we have

$$\begin{aligned}\mathbf{MAx}&=\mathbf{Mb}\\ \mathbf{Ux}&=\mathbf{Mb}\end{aligned} \tag{26}$$

We can use the “backward process” and solve the linear system of equations using only determinants.

For a matrix **A** with floating point entries, this process requires floating point multiplications; the number of multiplications is

$$\frac{n(n-1)(2n-1)}{6}+\frac{(n-2)(n-1)}{2}=\frac{n^3}{3}-\frac{4}{3}n+1 \tag{27}$$

3.2 New Decomposition Theorem with Determinants and Total Pivoting

Gauss elimination in real numbers is unstable due to the possibility of finding arbitrarily small pivots. This process can be alleviated, however, by exchanging rows during the elimination. In our case, we exchange a row or column in a similar way to the Gauss process only when a particular pivot is zero. The following theorem is given without proof.

Theorem. Let $\mathbf{A}\in\mathbf{R}^{\mathbf{nxn}}$. Suppose that the New Transformation $\mathbf{M}_1^{'}\cdots\mathbf{M}_{\mathbf{k-1}}^{'}$, row permutation matrices $\mathbf{P}=\mathbf{P}_1\cdots\mathbf{P}_{\mathbf{k-1}}$ and column permutation matrices $\mathbf{\Pi}=\mathbf{\Pi}_1\cdots\mathbf{\Pi}_{\mathbf{k-1}}$ have been determined such that $\mathbf{U}=\mathbf{M}_{\mathbf{k-1}}^{'}\mathbf{P}_{\mathbf{k-1}}\cdots\mathbf{M}_1^{'}\mathbf{P}_1\mathbf{A}\mathbf{\Pi}_1\ldots\mathbf{\Pi}_{\mathbf{k-1}}$. Then, the Upper Matrix **U** is obtained from $\mathbf{PA\Pi}$ without exchanging any rows or columns and $\mathbf{U}=\mathbf{MPA\Pi}$. Furthermore, if *exch* ≡ *the number of row exchanges plus the number of column exchanges*, we have $|\mathbf{A}|=(-1)^{exch}u_{n,n}$.

3.3 Matrix L and U

Now, although it is not strictly necessary, should we wish to obtain the matrix **L**, then the elements of the matrices **L**=(l_{ij}) and **U**=(u_{ij}) can be computed starting from the following formulas:

$$\begin{aligned}
&u_{1,k}=a_{1,k} \qquad k=1,\dots,n\\
&l_{j,1}=\frac{a_{j,1}}{u_{1,1}} \qquad j=2,\dots,n\\
&u_{jk}=\left(a_{jk}-\sum_{s=1}^{j-1}l_{js}u_{sk}\right)u_{j-1\,j-1} \ ;\ k=j,\dots,n \ ;\ j\ge 2\\
&l_{jk}=\frac{1}{u_{kk}}\left(a_{jk}-\sum_{s=1}^{k-1}l_{js}u_{sk}\right) \ ;\ j=k,\dots,n \ ;\ k\ge 2
\end{aligned} \tag{28}$$

4 Test Problems and Numerical Results

We use the combinatorial matrix [13] whose mathematical expression is:

$$A=\left(y+\delta_{ij}x\right);\quad \delta_{ij}=1 \quad \forall \quad i=j;$$

$$\delta_{ij}=0 \quad \forall \quad i\ne j \ ;\quad b=\begin{pmatrix} ny+x\\ \cdot\\ \cdot\\ \cdot\\ ny+x\end{pmatrix}; \tag{29}$$

and whose determinant and adjoint matrix are expressed as

$$|A|=x^{n-1}(x+ny)$$

$$A^{Adj}=\begin{pmatrix} x^{n-2}[x+(n-1)] & -x^{n-2}y & \cdots & -x^{n-2}y\\ -x^{n-2}y & x^{n-2}[x+(n-1)] & \cdots & -x^{n-2}y\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ -x^{n-2}y & -x^{n-2}y & \cdots & x^{n-2}[x+(n-1)]\end{pmatrix}. \tag{30}$$

This section presents the performance data for the algorithm of the new decomposition described in Section 3, indicating the result obtained with the last value of the diagonal of the lower matrix which represents the determinant. The results are given in Figure 1.

4.1 Tests with Mathematica Software and PC HP Pavilion Quadcore

The numerical tests for GT on an array of size 1020x1020 and $x = 2, y = 1$ yielded the expected result of 1021 with a CPU time of 2.371 seconds and the number of processors used was 4. Version 8 of Mathematica software was used. Another test for GT on an array of size 1430x1430 and $x = 2, y = 1$ yielded the expected result of 1431 with a CPU time of 5.32 seconds.

The architecture used for this example is a quad-core Intel (R) Core (TM) i7 CPU Q720@1.60GHz , installed memory(RAM): 6.00 GB, system type: 64-bit Operating System, model: HP Pavilion dv7 Notebook PC.

4.2 Tests with GNU Octave Software and Cluster Beowolf Supermicro

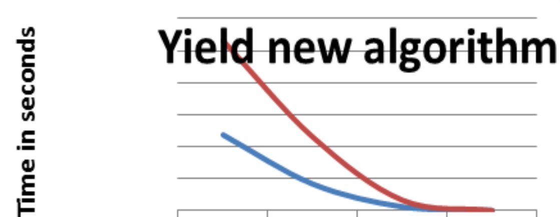
The Det() routine yielded the expected result of 1021, with a CPU time of 0.830 seconds and the number of processors used was 1. Version 3.0.5 of software GNU Octave on OpenMP was used, and the other result obtained was 1431, with a CPU time of 2.3 seconds.

Another test for GT on an array of size 1500x1500 and $x = 2, y = 1$ yielded the expected result of 1501.218, with a CPU time of 0.173 seconds and the number of processors used was 1. ACML-SGESVof software optimized routine for AMD processors was used.

These examples were obtained on a cluster Beowolf Supermicro, with 7 nodes with 4 AMD Quad-core each one = 7 x16 = 112 cores; 32 Gb RAM by Node = 224 Gb; OS Linux (a free version Red Hat) CentOS 5. Clusters were connected by 2 networks: Gigabit and Infiniband (20Gb).

4.3 Tests with Version 1 of Code Programmed in C lenguage on a GeForce GTX 480 with CUDA Driver Version 3.2

The numerical tests for GT on an array of size 1020x1020 and $x = 2, y = 1$ yielded the expected result of 1021, with a CPU time of 0.119 seconds and the number of processors used was 480.



	PC HP PAVILION	SUPER MICRO BEOWOLF	GeForce GTX 480	TeslaC2050
N=1020	2.371	0.83	0.119	0.001
N=1430	5.32	2.3	0.323	0.001

Fig. 1. The obtained **r**esults

Version 1 of code programmed in C language software was used. Another test with an array of size 1430x1430 and $x = 2, y = 1$ yielded the expected result of 1431, with a CPU time of 0.323 seconds.

These examples were obtained on a GeForce GTX 480 with CUDA Driver Version 3.20, 480 Cores, total amount of global memory 1609 GB, Clock rate 1.40 GHz, SINGLE PRECISION, on a platform Linux Debian 6.0.

Besides, we also used the library routine CULAS: SGESV to compare its efficiency in calculation, and the results presented in Fig. 1 were obtained with the CPU time of 1 second.

4.4 Tests with Version 1 of Code Programmed in C Language on a Tesla C2050 with CUDA Driver Version 4.20

Finally, we used the examples described previously in a Tesla C2050 with CUDA Driver Version 4.20, 448 CUDA cores, total amount of global memory 2817 GB, clock rate 1.15 GHz, SINGLE PRECISION, and obtained the expected result of 1021, with a CPU time of 0.001 seconds.

Version 1 of code programmed in C language on a platform Linux Ubuntu™ version 11.04 was used. Another test with an array of size 1430x1430 and $x = 2, y = 1$ yielded the expected result of 1431, with a CPU time of 0.001 seconds.

5 Conclusions

In this paper we introduced a new theorem on decomposition into determinants of matrix **A** and new linear transformations expressed as Equations 11, 12 and 28.

The majority of simultaneous linear equation systems can also be solved with these new linear tranformations. The result is Cramer-type solutions of $O(n^3)$. This fact is new.

We also proposed a modified Doolittle-Gauss-LU-Decomposition in two versions: the first one was applied to matrix **A** and the second one to the augmented matrix **(A|b)**.

The first one is a new algorithm to compute determinants in exact form if and only if $\mathbf{A} \in \mathbf{I}^{\mathbf{n}}$, and the second is a new LU elimination process to solve linear system of equations in parallel massive form using multicore systems with accelerated GPU [14].

Finally, when referring to Cramer's rule, it was affirmed by G. Strang [15] that: "…Thus each component of *x* is a ratio of two determinants, a polynomial of degree *n* divided by another polynomial of degree *n*. This fact might have been recognized from Gauss elimination, but it never was". This is made evident in the present paper.

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