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A Finite-Time Consensus Algorithm with Simple Structure for Fixed Networks

David Gómez-Gutiérrez[1], Javier Ruiz-León[2], Sergej Čelikovský[3] [4], Juan Diego Sánchez-Torres[5]

[1] Tecnológico de Monterrey, Campus Guadalajara, Jalisco, Mexico

[2] CINVESTAV-IPN, Unidad Guadalajara, Jalisco, Mexico

[3] Institute of Information Theory and Automation, Czech Academy of Sciences, Prague, Czech Republic

[4] Czech Technical University in Prague, Faculty of Electrical Engineering, Prague, Czech Republic

[5] ITESO Guadalajara, Department of Mathematics and Physics, Jalisco, Mexico

david.gomez.g@ieee.org, celikovs@utia.cas.cz, {dsanchez, jruiz}@gdl.cinvestav.mx

Abstract. In this paper, a continuous-time consensus algorithm with guaranteed finite-time convergence is proposed. Using homogeneity theory, finite-time consensus is proved for fixed topologies. The proposed algorithm is computationally simpler than other reported finite-time consensus algorithms, which is an important feature in scenarios of energy efficient nodes with limited computing resources such as sensor networks. Additionally, the proposed approach is compared on simulations with existing consensus algorithms, namely, the standard asymptotic consensus algorithm and the finite-time and fixed-time convergent algorithms, showing, in cycle graph topology, better robustness features on the convergence with respect to the network growth with less control effort. Indeed, the convergence time of other previously proposed consensus algorithms grows faster as the network grows than the one herein proposed whereas the control effort of the proposed algorithm is lower.

Keywords. Finite-time consensus, multi-agent systems, multiple interacting autonomous agents, self-organizing systems.

1 Introduction

Motivated by the social insects' ability to self-organize and mutually cooperate relying only on neighbor-to-neighbor communication, there has been an increasing interest during the last decade in the distributed algorithms obtaining a desired global behavior from local interactions. One of such algorithms is the consensus algorithm [15], which allows a swarm to agree on a common value in a distributed fashion (see e.g. [8,9,17,18,21,27]), using only communication among its neighbors. One of the possible applications of the consensus that achieved recently a special interest is the application to sensor networks [8]. In this scenario, of energy efficient nodes with limited computing resources, computationally simple algorithms for self-organization are of paramount importance.

It has been shown that if the underlying graph topology is strongly connected then consensus can be achieved [20]. Moreover, if the graph is balanced (identical number of in-neighbors and out-neighbors), then asymptotic convergence to the average of the swarm's initial value is obtained by the standard consensus algorithm [20]. For graphs that are not balanced the algorithm can be modified to still achieve average consensus whenever each node has knowledge on the number of its out-neighbors [4], and compensates accordingly using a surplus variable for each node.

These results have been extended, using nonlinear theory and scalar functions used in fixed-time and finite-time stability, to achieve finite-time [11, 25, 26, 28, 30], and fixed-time convergent consensus [31, 32], (i.e. there exists a bound for the convergence time independent of the initial conditions [10, 22]). However, as we illustrate later, the convergence time does not uniquely depend on the initial conditions but mainly on the network topology, where the convergence time grows as the algebraic connectivity of the graph decreases.

In this paper, a finite-time convergence consensus algorithm for fixed networks is proposed. As a matter of fact, it was claimed in [28], that the consensus algorithm analyzed in the present paper could later on be justified as a consensus algorithm. Nevertheless, neither full rigorous proof, nor its sketch were provided. It is the aim of the paper to formally prove that such algorithm is a consensus algorithm for fixed networks, which is achieved using homogeneity theory [7, 14, 23]. The hereinafter proposed algorithm is shown to be computationally simpler than other finite-time consensus algorithms ([11, 16, 28, 31]). Namely, in the proposed approach, each node requires a single computation of the nonlinear term, while in ([11, 16, 28, 31]), the number of computations of the nonlinear term in each node equals the number of nodes adjacent to it.

Moreover, we illustrate through simulations that the proposed algorithm shows interesting robustness properties, in networks with cycle graph topology, with respect to the network growth with less control effort, namely that it shows more robustness to the network growth than previously proposed consensus algorithms. In particular, we compare our results with other consensus algorithms, namely, the asymptotic consensus [20], finite-time consensus [28] and fixed-time convergent consensus [31] algorithms.

The paper is organized as follows. In Section 2, we present the mathematical preliminaries together with some previously proposed consensus algorithms. In Section 3, the main results are presented, followed by an illustrative example in Section 4. Finally, in Section 5, the conclusions and the future work are presented.

2 Preliminaries

2.1 Finite-Time Stability

We recall in this section some results on homogeneity theory [7, 14, 23] and finite-time stability [1, 2], that will be used later on. Further facts on finite time stability can be found in [2, 13, 19, 24].

Definition 1. *[1] Consider the nonlinear system:*

$$\dot{x} = f(x) \quad \textit{with} \quad f(0) = 0. \tag{1}$$

Here, $f \in \mathcal{C}^1(\mathcal{D} \setminus \{0\} \cap \mathcal{C}^0(\mathcal{D}))$ where $\mathcal{D} \subset \mathbb{R}^n$ is some region in R^n, i.e. an open and connected subset of $\mathbb{R}^n$. Let $\psi(t, x_0)$ and $T_m(x_0) \in (0, \infty)$ be such that $\forall t \in [0, T_m(x_0)), \forall x_0 \in \mathcal{D} \setminus \{0\}$ such that:

$$\frac{\mathrm{d}\psi(\mathrm{t}, \mathrm{x}_0)}{\mathrm{d}t} = f(\psi(t, x_0)), \quad \psi(0, x_0) = x_0,$$

where $T_m(x_0)$ is a maximal possible real number with the above property, or plus infinity, such $T_m(x_0)$ and $\psi(t, x_0)$ always exist and are unique by $f \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\})$.

The origin is called as finite-time convergent for (1) *if there exists an open neighborhood $\mathcal{N} \subseteq \mathcal{D}$ of the origin that is forward invariant with respect to* (1) *and a function $T : \mathcal{N} \setminus \{0\} \to (0, \infty)$, called the settling-time function, such that for every $x_0 \in \mathcal{N} \setminus \{0\}$, the solution $\psi(t, x_0)$ is defined on $[0, T(x_0))$, $\psi(t, x_0) \in \mathcal{N} \setminus \{0\}$ for all $t \in [0, T(x_0))$ and $\lim_{t \to T(x_0)} \psi(t, x_0) = 0$. Furthermore, it is called finite-time stable if it is stable and finite-time convergent, and globally finite-time stable if it is finite-time stable with $\mathcal{D} = \mathcal{N} = \mathbb{R}^n$.*

Note that by the uniqueness of $\psi(t, x_0)$, it follows that $T(x_0) = \min\{t \in \mathbb{R}_+ : \phi(t, x_0) = 0\}$ for all $x_0 \in \mathcal{N} \setminus \{0\}$.

Definition 2. *A function $V : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree l with respect to the "standard dilation":*

$$\Delta_\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n), \tag{2}$$

if and only if:

$$V(\lambda x_1, \ldots, \lambda x_n) = \lambda^l V(x_1, \ldots, x_n),$$

for all $\lambda > 0$.

Definition 3. *A vector field $f(x)$, $x \in \mathbb{R}^n$ is homogeneous of degree q with respect to the standard dilation (2) if and only if the i-th component f_i is homogeneous of degree $q+1$ with respect to (2), i.e. $f_i(\lambda x_1, \ldots, \lambda x_n) = \lambda^{q+1} f(x_1, \ldots, x_n)$, $\lambda > 0$, $i = 1, \ldots, n$.*

Theorem 4. *[2, Theorem 7.1] Let $f(x)$, $x \in \mathbb{R}^n$ be a homogeneous vector field of degree q with respect to (2). Then the origin of (1) is finite-time stable if and only if it is asymptotically stable and $q < 0$.*

Lemma 5. *[2] The right-hand side of:*

$$\dot{y} = -k\lfloor y \rceil^\alpha, \quad \alpha \in (0, 1), \tag{3}$$

where $\lfloor y \rceil^\alpha = |y|^\alpha \operatorname{sign}(y)$, is homogeneous of degree $\alpha - 1$ with respect to the standard dilation. Thus, (3) is finite-time stable and its settling-time is given by $T(y_0) = \frac{1}{k(1-\alpha)}|y_0|^{1-\alpha}$.

Lemma 6. *[29, 32] Consider the non-homogeneous system:*

$$\dot{y} = -k_1\lfloor y \rceil^{2-p/q} - k_2\lfloor y \rceil^{p/q}, \tag{4}$$

then the origin of (4) *is finite-time stable and uniformly convergent (i.e. the bound on the convergence time is independent of the initial conditions) and its settling-time is bounded by $T \leq \frac{q\pi}{2\sqrt{k_1 k_2}(q-p)}$.*

2.2 Graph Theory

The following notation and preliminaries on graph theory are taken mainly from [12].

A graph $\mathcal{X}$ consists of a vertex set $\mathcal{V}(\mathcal{X})$ and an edge set $\mathcal{E}(\mathcal{X})$, where an edge is an unordered pair of distinct vertices of $\mathcal{X}$. We write ij to denote an edge and say that the vertex i and vertex j are adjacent or that j is a neighbor of i and denote this by $j \sim i$. A weighted graph is a graph together with a weight function $W : \mathcal{E}(\mathcal{X}) \to \mathbb{R}_+$ on its edges. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ of a graph with n vertices is a square matrix where a_{ij} corresponds to the weight of the edge ij. In this paper we only consider undirected graphs and therefore $a_{ij} = a_{ji}$. The Laplacian of $\mathcal{X}$ is $\mathcal{Q}(\mathcal{X}) = \Delta - A$ where $\Delta = diag(d_1 \cdots, d_n)$ with $d_i = \sum_{j=1}^n a_{ij}$. For undirected graphs the Laplacian matrix $\mathcal{Q}$ is a positive semidefinite and symmetric matrix, thus its eigenvalues are all real and non-negative. If the graph $\mathcal{X}$ is connected then the eigenvalue $\lambda_1(\mathcal{Q}) = 0$ has algebraic multiplicity one with eigenvector $\mathbb{1} = [1 \ \cdots \ 1]^T$.

For the Laplacian $\mathcal{Q}(\mathcal{X})$ there exists a factorization $\mathcal{Q}(\mathcal{X}) = DD^T$ (D is known as the incidence matrix of $\mathcal{X}$ [12]) where D is a $|\mathcal{V}(\mathcal{X})| \times |\mathcal{E}(\mathcal{X})|$ matrix (where $|\mathcal{V}(\mathcal{X})|$ and $|\mathcal{E}(\mathcal{X})|$ are the cardinality of the vertex and edge set, respectively), such that if $ij \in \mathcal{E}(\mathcal{X})$ is an edge with weight a_{ij} then the corresponding column of D has only two nonzero elements with the $i-$th element $a_{ij}^{\frac{1}{2}}$ and the $j-$th element $-a_{ij}^{\frac{1}{2}}$.

A path from i to j in a graph is a sequence of distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of the graph $\mathcal{X}$ then $\mathcal{X}$ is connected. Through this work we consider only connected graphs.

2.3 Asymptotic, Finite-Time and Fixed-Time Convergent Consensus

In this subsection we present the standard consensus algorithm with asymptotic convergence and describe how nonlinear algorithms are derived from it, with particular focus on finite-time and fixed-time convergent algorithms.

Let $\mathcal{X}$ be a dynamic network. The following equation defines the consensus algorithm that follows each agent:

$$\dot{x}_i = u_i, \tag{5}$$

where $x_i \in \mathbb{R}$ is the dynamic of the $i-th$ agent and u_i defines the different consensus laws. For instance, the standard consensus algorithm with asymptotic convergence proposed in [20] is given by:

$$u_i = k \sum_{j \in \{j : ji \in \mathcal{E}(\mathcal{X})\}} a_{ij}(x_j - x_i), \tag{6}$$

where $a_{ij} \geq 0$.

Considering nonlinear functions $f(x)$, $f(0) = 0$, such that the origin of $\dot{x} = -f(x)$ is stable, two directions have been taken to derive nonlinear consensus algorithms from (6). On the one hand with the algorithm:

$$u_i = k \sum_{j \in \{j : ji \in \mathcal{E}(\mathcal{X})\}} a_{ij} f(x_j - x_i), \tag{7}$$

and on the other hand with the algorithm:

$$u_i = kf\left(\sum_{j \in \{j : ji \in \mathcal{E}(\mathcal{X})\}} a_{ij}(x_j - x_i)\right). \tag{8}$$

There by, the consensus algorithms proposed in [28] and [32] are derived from (3) and (4), respectively, following the direction (7). Namely, the finite-time consensus algorithm proposed in [28] is:

$$u_i = k \sum_{j \in \{j : ji \in \mathcal{E}(\mathcal{X})\}} a_{ij} \lfloor x_j - x_i \rceil^{\alpha}, \quad \alpha \in (0, 1). \tag{9}$$

The fixed-time convergent algorithm proposed in [31] for fixed topologies is:

$$u_i = \sum_{j \in \{j : ji \in \mathcal{E}(\mathcal{X})\}} a_{ij}(k_1 \lfloor x_j - x_i \rceil^{2-\frac{p}{q}} + k_2 \lfloor x_j - x_i \rceil^{\frac{p}{q}}), \tag{10}$$

where $k_{1,2} > 0$, and p, q are positive odd numbers.

Although, both [28] and [32] ([28] does not present the proof of the claim), address the consensus derived from (3) and (4), respectively, following also the direction (8), the results in both papers are restricted to static networks, i.e. where $\mathcal{X} = \mathcal{X}$ for all $t \geq t_0$. In the following section we prove that the algorithm using functions (3) and direction (8) achieves finite-time consensus over fixed networks and illustrate by simulations an interesting property on the robustness of the convergence time with respect to the growth of the network. In particular, we show that the convergence time to the consensus state grows faster as the network grows (as the smallest nonzero eigenvalue decreases), in (9) an (10) than in the proposed algorithm.

3 Main Results

In this section we derive the mathematical proofs to show the finite-time convergence to the consensus state of the proposed algorithm under fixed networks. First, we show using Lyapunov theory, the asymptotic convergence to the consensus state. Then, the finite-time convergence follows by using homogeneity [5–7, 14, 23] and Theorem 4.

The aim of this paper is to show that, if $\mathcal{X}$ is a connected graph, then the algorithm $\dot{x}_i = u_i$, with:

$$u_i = k \lfloor \sum_{j \in \{j : ji \in \mathcal{E}(\mathcal{X})\}} a_{ij}(x_j - x_i) \rceil^{\alpha}, \quad \alpha \in [0, 1), \tag{11}$$

where $k > 0$, $\lfloor \bullet \rceil^{\alpha} = |\bullet|^{\alpha} \operatorname{sign}(\bullet)$ for $\alpha \in (0, 1)$ and $\lfloor \bullet \rceil^{0} = \operatorname{sign}(\bullet)$, achieves consensus with finite-time convergence.

Remark 7. *Notice that using the proposed consensus algorithm* (11)*, a node requires a single computation of the nonlinear function $f(\cdot)$, while in the consensus algorithms* (9) *and* (10) *proposed in [28] and [32], respectively, the number of nonlinear operations of each node equals the number of nodes adjacent to it. Thus, the consensus algorithm* (11) *is, in general, computationally less expensive than previously proposed finite-time consensus algorithms, namely [28] and [31].*

Remark 8. *In [28] it was shown that* (9)*, is a consensus algorithm with finite-time convergence. The authors claimed (without proof), that using their framework, it can be shown that for "static networks"* (11)*, is also a consensus algorithm with finite-time convergence. In the following we provide a rigorous proof, using Lyapunov theory and homogeneity theory [14,23], to show that* (11)*, is a consensus algorithm with finite-time convergence in fixed communication networks.*

Let $e = [e_1, \dots, e_n]^T$, where $e_i = \sum_{j \in \{j: ji \in \mathcal{E}(\mathcal{X})\}} (x_j - x_i)$, and notice that $e = -\mathcal{Q}(\mathcal{X}_{\mathcal{X}(t)})x$, where $x = [x_1, \dots, x_n]^T$ and therefore the network dynamics is:

$$\dot{x} = -F(\mathcal{Q}(\mathcal{X})x) = F(e), \tag{12}$$

where

$$F(e) = \begin{bmatrix} k\lfloor e_1 \rceil^\alpha \\ \vdots \\ k\lfloor e_n \rceil^\alpha \end{bmatrix}.$$

Thus, the dynamic for $e(t)$ is:

$$\dot{e} = -\mathcal{Q}(\mathcal{X})F(e), \tag{13}$$

where $\mathcal{Q}(\mathcal{X})$ is the graph Laplacian of $\mathcal{X}$.

In the following, (13) will be referred as the error dynamics. The i-th entry of $F(e)$ is denoted by $f(e_i)$, i.e. $f(e_i) := k\lfloor e_i \rceil^\alpha$.

Lemma 9. *Let $f(e_i) = \lfloor e_i \rceil^\alpha$, $\alpha \in [0, 1)$. Then:*

$$-\left(f(e_i) - f(e_j)\right)(e_i - e_j) \le 0.$$

In the case when $\alpha \in (0, 1)$ the equality holds iff $e_i - e_j = 0$ whereas for the case when $\alpha = 0$ the equality holds iff $\operatorname{sign}(e_i) = \operatorname{sign}(e_j)$.

Proof. Straightforward. □

Proposition 10. *Let $\mathcal{X}$ be a connected graph and let $\alpha \in [0, 1)$. Then, the origin of:*

$$\dot{e} = -\mathcal{Q}(\mathcal{X})F(e), \tag{14}$$

is globally asymptotically stable.

Proof. Consider the candidate Lyapunov function $V(e) = \frac{1}{2} \sum_{i \in \mathcal{V}(\mathcal{X})} e_i^2$, then its time derivative along the trajectories of (14) is:

$$\dot{V}(e) = \frac{1}{2} \sum_{i \in \mathcal{V}(\mathcal{X})} e_i \dot{e}_i = -e^T \mathcal{Q}(\mathcal{X})F(e).$$

Then according to Subsection 2.2, there exists a factorization for the graph Laplacian $\mathcal{Q}(\mathcal{X})$, such that, $\mathcal{Q}(\mathcal{X}) = D_k D_k^T$. Thus, $\dot{V}(e) = -e^T D_k D_k^T F(e)$. Therefore, $ij \in \mathcal{E}(\mathcal{X})$, implies that the entry of $e^T D_k$ corresponding to the edge ij is $a_{ij}^{\frac{1}{2}}(e_i - e_j)$; in the same way, the entry of $D_k^T F(e)$ corresponding to the edge ij is $a_{ij}^{\frac{1}{2}}(f(e_i) - f(e_j))$ [12], therefore due to Lemma 9 we have that:

$$\dot{V}(e) = -\sum_{ij \in \mathcal{E}(\mathcal{X}), i > j} a_{ij}(f(e_i) - f(e_j))(e_i - e_j) \le 0, \tag{15}$$

where the equality in (15), for the case when $\alpha = 0$, holds if $\operatorname{sign}(e_i) = \operatorname{sign}(e_j)$, for each $ij \in \mathcal{E}(\mathcal{X}_k))$, which implies, since the graph $\mathcal{X}$ is connected, that $\operatorname{sign}(e_1) = \cdots = \operatorname{sign}(e_n)$. In a similar way, the equality in (15), for the case when $\alpha \in (0, 1)$, holds iff $e_i - e_j = 0$ for each $ij \in \mathcal{E}(\mathcal{X}_k))$, which implies, since the graph $\mathcal{X}$ is connected, that $e_1 = \cdots = e_n$.

However, since $e = -\mathcal{Q}(\mathcal{X})x$ and $1 \in \ker(\mathcal{Q}(\mathcal{X}))$, then $1^T e = -1^T \mathcal{Q}(\mathcal{X})x = 0$ i.e. $\sum_{i \in \mathcal{V}(\mathcal{X})} e_i = 0$ and therefore $\operatorname{sign}(e_1) = \cdots = \operatorname{sign}(e_n)$ and $e_1 = \cdots = e_n$ can only hold iff $e = 0$. Thus, $\dot{V}(e) \le 0$ with $\dot{V}(e) = 0$ iff $e = 0$. Therefore, the origin of (13) is globally asymptotically stable. □

Lemma 11. *Let $\alpha \in (0, 1)$ and let $H(e)$ be the right hand side of (13), i.e. $H(e) := -\mathcal{Q}(\mathcal{X})F(e)$. Then, the vector field H is homogeneous of degree $\alpha - 1$ with respect to the standard dilation.*

Proof. Notice that the function $f(e_i)$ is homogeneous of degree α with respect to the standard dilation, i.e. for $\lambda > 0$:

$$\begin{aligned} f(\lambda e_i) &= k|\lambda e_i|^\alpha \operatorname{sign}(\lambda e_i), \\ &= k(|\lambda||e_i|)^\alpha \operatorname{sign}(e_i), \\ &= \lambda^\alpha k|e_i|^\alpha \operatorname{sign}(e_i), \\ &= \lambda^\alpha f(e_i). \end{aligned}$$

Now, let the i-th component of the vector field $H(e)$ be denoted as $H^i(e)$, then $H^i(e) = \sum_{j \in \{j: ji \in \mathcal{E}(\mathcal{X})\}} (f(e_j) - f(e_i))$ and:

$$\begin{aligned} H^i(\lambda e_i) &= \sum_{j \in \{j: ji \in \mathcal{E}(\mathcal{X})\}} (f(\lambda e_j) - f(\lambda e_i)), \\ &= \sum_{j \in \{j: ji \in \mathcal{E}(\mathcal{X})\}} (\lambda^\alpha f(e_j) - \lambda^\alpha f(e_i)), \\ &= \lambda^\alpha H^i(e_i). \end{aligned}$$

Since $r_i = 1$, $i = 1, \dots, m$, then $q = \alpha - 1$ and $H(e)$ is homogeneous of degree $\alpha - 1$ with respect to the standard dilation. □

The main result of this paper is the following.

Theorem 12. *Let $\mathcal{X}$ be a connected graph. Then, the algorithm (11) achieves consensus on a fixed network in finite-time.*

Proof. According to Proposition 10, the dynamic of $e(t)$ is asymptotically stable. Moreover, since the graph $\mathcal{X}$ is connected, $e = -\mathcal{Q}(\mathcal{X})x = 0$ implies that $x \in span(1)$, i.e. that consensus is achieved. The finite-time stability for the case when $\alpha = 0$ follows by noticing that if the i-th agent is the one such that $x_i(t_0) = x_{max}(t_0) = max(x_1(t_0), \ldots, x_n(t_0))$ then u_i is $u_i = -k$ as long as $e \neq 0$. In a similar way, if $x_j(t_0) = x_{min}(t_0) = min(x_1(t_0), \ldots, x_n(t_0))$ then u_i is $u_i = k$ as long as $e \neq 0$. Thus, the consensus value, with $\alpha = 0$ is $\bar{x} = \dfrac{x_{max}(t_0) + x_{min}(t_0)}{2}$ and the convergence time is $t_{reach} = \dfrac{x_{max}(t_0) - \bar{x}}{k}$. Moreover, the finite time stability for the case when $\alpha \in (0, 1)$ follows from homogeneity theory, since the vector field $H(e)$ is homogeneous of degree $q = \alpha - 1 < 0$, then according to Theorem 4, (13) is finite-time stable. Hence consensus is achieved with finite-time convergence in a fixed topology. □

Remark 13. *Notice that the convergence time for the case when $\alpha = 0$ is independent of the underlying connected graph topology, and thus independent of the network size.*

4 An Illustrative Example Showing Robustness to the Network Growth

In this section we illustrate that, in networks with cycle graph topology, the convergence of the proposed consensus algorithm (11), is more robust to the network growth than the standard consensus algorithm (6) proposed in [20], the finite-time consensus algorithm (9) proposed in [28] and the fixed-time convergent algorithm (10) proposed in [31] while requiring less control effort. To this end, we consider a communication topology described by an undirected cycle graph $\mathcal{C}_n$, where $\mathcal{C}_n$ is a cycle graph with vertex set $\{0, \ldots, n-1\}$ such that vertex i is adjacent to vertex j, i.e. $j \sim i$, if and only if $j - i \equiv \pm 1 (\mod n)$.

We simulate three different scenarios to illustrate the convergence of the above mentioned consensus algorithms with respect to the network growth (the simulation was performed using the Modelica® language and simulated in Dymola® using the Euler integration method). Namely, a network formed by an undirected cycle graph $\mathcal{C}_{25}$, $\mathcal{C}_{200}$, and $\mathcal{C}_{1000}$, respectively.

We would like to highlight that the initial conditions are set the same for the different algorithms using the linear congruential generator [3]:

$$z_{i+1} = r z_i + s \ (\mod M) \ n \geq 0, \ x_i(t_0) = l \frac{z_i}{M} - m,$$

where $r = 45$, $s = 1$, $M = 1024$, $l = 20$ and $m = 10$, which produces a pseudo–random sequence of initial conditions $x_i(t_0)$ in the interval $[-10, 10]$. The parameters of the different algorithms are set experimentally, according to Table 1, such that, in the 25 agents network $\mathcal{C}_{25}$, each algorithm achieves 99% of its final consensus value approximately at 2 seconds.

Table 1. Parameters selected for the different algorithms

Algorithm	Parameters
Proposed Algorithm	$k = 7$ and $\alpha = 0.5$
Asymptotic Consensus (6)	$k = 35$
Finite-time Consensus (9)	$k = 12$ and $\alpha = 0.5$
Fixed-time Consensus (10)	$k_1 = k_2 = 8.5$ and $p/q \approx 0.5$

The evolution of the above mentioned algorithms for a cycle graph $\mathcal{C}_{25}$ is presented in Fig. 1, where it can be seen that each algorithm achieves 99% of its final consensus value approximately at 2 seconds, as explained before. At this point, considering the convergence of the algorithms, no true advantages are identified from one another. For instance, even if the convergence of the algorithm in [20], is asymptotic, the 99% of the consensus value is achieved approximately at the same time as the algorithm in [28], which has a finite-time convergence. Moreover, even though in [31], there exists a bound for the convergence time, independent from the initial conditions, in many applications as in wireless sensor networks, the initial conditions are within a known range and such convergence bound also depends on the network topology which is usually unknown a priori.

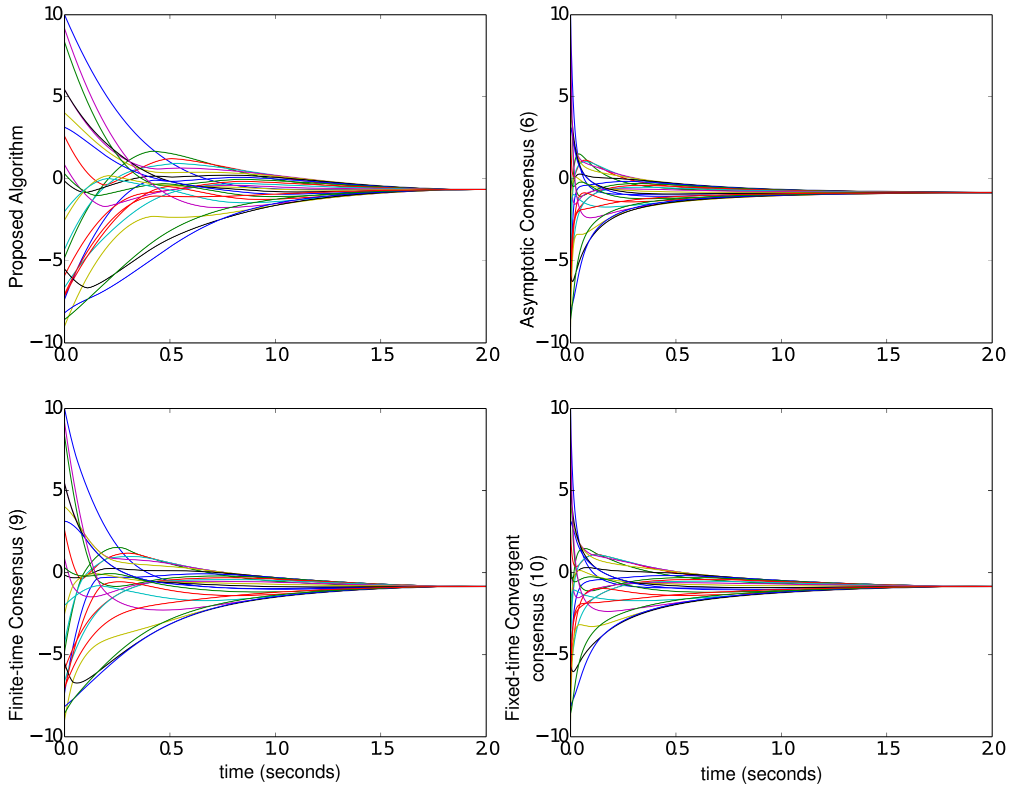

Fig. 1. Different consensus algorithms experimentally set for convergence to the consensus state approximately at 2 seconds under a cycle graph $\mathcal{C}_{25}$ communication topology

Table 2. Total energy comparison among the different algorithms

Algorithm	Total Energy E_{tot}
Proposed Algorithm	26.44
Asymptotic Consensus (6)	83.00
Finite-time Consensus (9)	37.49
Fixed-time Consensus (10)	64.21

Now, let us consider the control effort related to each agent as $E_i = \left(\int_{t_0}^{t_1} u_i^2\right)^{\frac{1}{2}}$ and the total control effort of the network as $E_{tot} = \sum_1^n E_i$ where n is the number of agents. Then, the control effort of the network E_{tot} of the experiment in Fig. 1 is presented in Table 2, showing that the proposed algorithm requires less control effort E_{tot} among the four algorithms to achieve consensus.

However, as the number of agents under a cycle graph communication topology grows (i.e. as the smallest nonzero eigenvalue decreases), the convergence time to the consensus state varies among the different algorithms.

For instance, when it grows from 25 to 200 agents the convergence of the asymptotic consensus algorithm grows from 2 seconds to approximately 120 seconds, as shown in Fig. 2. The consensus of the finite-time consensus of [28] grows to 20 seconds, the fixed-time convergent one to 30 seconds whereas the proposed consensus algorithm grows from 2 seconds to 7 seconds.

Moreover, as the cycle network grows to $\mathcal{C}_{1000}$, the convergence time grows to 10 seconds for the proposed algorithm, 550 seconds for the asymptotic consensus algorithm, 60 seconds for the finite-time consensus (9) and 70 seconds for the fixed-time convergent consensus algorithm (10) as shown in Fig. 3.

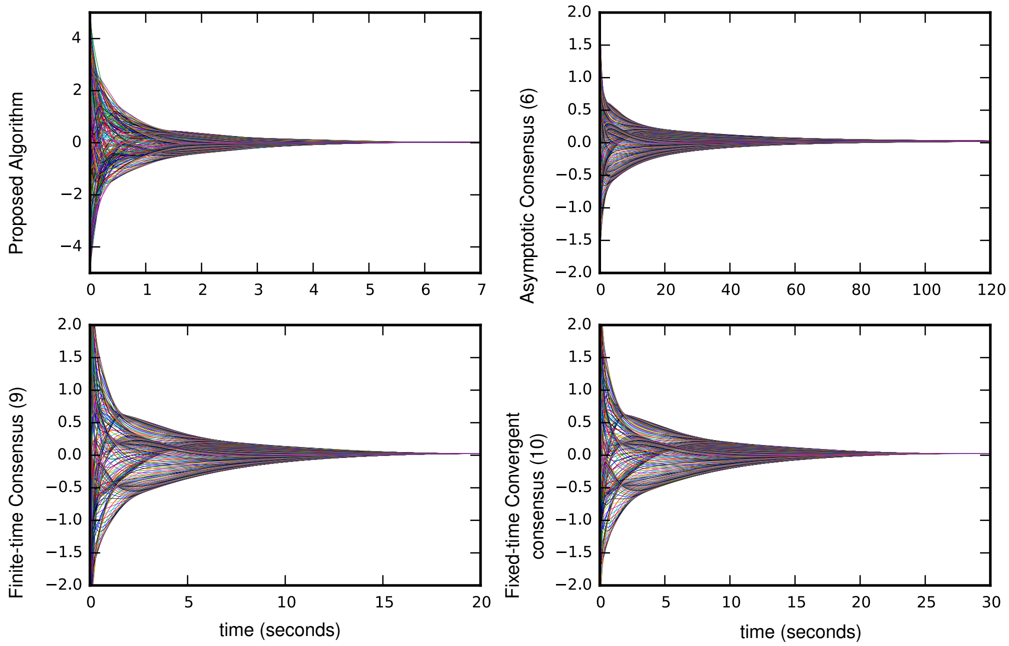

Fig. 2. Comparison among different consensus algorithms when the network grows to 200 agents with communication topology $\mathcal{C}_{200}$

It is worth noting that in many scenarios the network size is not determined a priori and in general it cannot be estimated by the agent to adjust accordingly.

Thus, algorithms that could be used over a wide range of scenarios (different network size and topology) with a single parameter configuration are desirable, contrary to algorithms whose convergence-time varies so widely that it becomes prohibited (as the asymptotic consensus algorithm of Fig. 2) in some applications.

5 Conclusions and Future Work

A continuous-time consensus algorithm with finite-time convergence over fixed networks was presented in this paper. Finite-time convergence was proven using homogeneity theory.

The proposed algorithm is computationally simpler than previously proposed finite-time consensus algorithms. Moreover, the proposed approach was compared, in simulations with existing consensus algorithms, including the standard consensus algorithm and finite-time and fixed-time convergent algorithms, showing its robustness, in a cycle graph topology, to the network growth.

Future work concerns the analysis of the proposed algorithm under dynamic networks which will be reported elsewhere.

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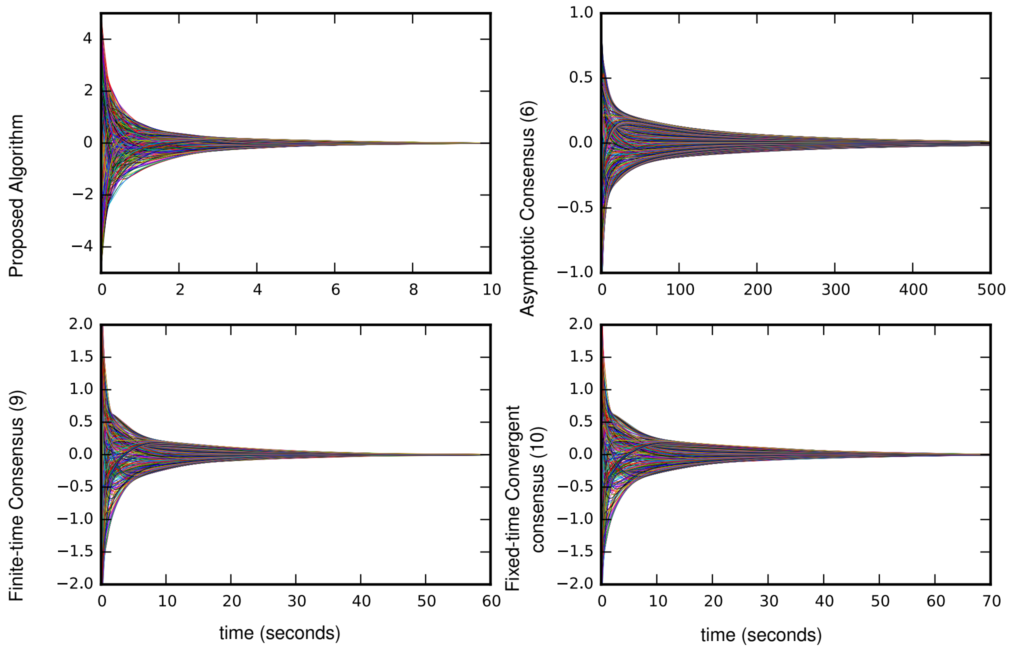

Fig. 3. Comparison among different consensus algorithms when the network grows to 1000 agents with communication topology $\mathcal{C}_{1000}$

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Corresponding author is David Gómez-Gutiérrez.