Computación y Sistemas Vol. 4 No. 2 pp. 85 - 93
 ISSN 1405-5546 Impreso en México

A Class of Parametric Regular Networks for Multicomputer Architectures[1]

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Article received on February 15, 2000; accepted on August 23, 2000

Abstract

A new class of multicomputer interconnection networks is proposed and analyzed: Parametrically described, Regular, and based on Semigroups (PRS) networks (or $R_s(N, v, g)$ graphs with the order N, the degree v, the girth g, and the number of equivalence classes s). The class of PRS networks includes many classes of known networks (hypercubes, circulant networks, cube-connected cycles, etc.) as special cases. We explore the basic topological properties (connectivity, isomorphism, lower bounds on the diameter and the average distance, etc.) of the proposed graphs and synthesize the optimal PRS networks having the minimal diameter for the given parameters of the graph. The PRS networks and their subclass - multidimensional circulants - are compared to hypercubes: the optimal PRS graph's diameter is $\approx 0.21 \log_2 N$ (for $g = 6$) and the circulant's diameter is $\approx 0.32 \log_2 N$ whereas the hypercube's diameter is $\log_2 N$, provided they have the same vertex and edge complexity.

Keywords: regular interconnection networks, parallel systems, circulant networks, hypercube topologies.

1 Introduction

A design of interconnection networks for parallel computer system architectures and distributed memory computer systems requires a study of undirected dense regular graphs with small diameters. Graphs with these properties can be found within the class of Cayley graphs, in particular in the class of circulant graphs (Du et al., 1990; Monakhova, 1991; Bermond et al., 1995) and also within the class of PRS graphs introduced in Monakhov (1979). The PRS graphs are a generalization of circulants, hypercubes, cube-connected cycles (Preparata and Vuillemin, 1981), chordal ring networks (Arden and Lee, 1981) and other classes of graphs used as interconnection networks of computer systems. In Kwai and Parhami (1996, 1998) Gaussian cubes are considered as generalization of hypercubes which also represent a subclass of the PRS graphs. Notice that hypercubes have a logarithmic estimate on the diameter only provided that a degree of a node grows with N as log N. The graphs from the class of PRS netwoks have a logarithmic estimate on the diameter (from N) for a fixed degree of a node of the graph in contrast to hypercubes. The other classes of graphs are presented in Scherson (1991) and Corbett (1992) with logarithmic estimate on the diameter.

2 $L(N, v, g)$ graphs

At first let us consider the class of $L(N, v, g)$ graphs introduced in Korneyev and Monakhov (1980) as a structure of parallel computer systems. These graphs are a basis for generation of the PRS graphs.

The $L(N, v, g)$ *graphs* are regular graphs of the degree v with the number of nodes N and the girth g. The algorithm for a construction of $L(N, v, g)$ graphs makes use of the notation of $L(v, g)$ graph which is an infinite planar graph of the degree v and the girth g. Note that a graph is called planar if it can be drawn at the plane

[1]This work is partially supported by RFBR grant N99-07-90422

so that any two of its edges are not intersected. The method of construction of $L(v,g)$ graphs is presented in Korneyev and Monakhov (1980). The example of fragment of $L(4,5)$ graph is shown in Figure 1.

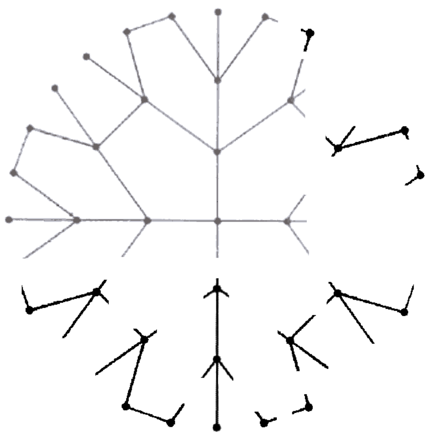

Figure Fragment of $L(4,5)$ graph

The network, belonging to the class of $L(N,v,g)$ graphs, will be formed:

(1) by the construction from the infinite planar graph $L(v,g)$ of a subgraph with the number of nodes N, forming the first $(k-1)$ layers and part of the kth layer of required network: $\sum_{i=0}^{k-1}|A_i| \le N \le \sum_{i=0}^{k}|A_i|$, where A_i is a set of nodes on the layer i of $L(v,g)$ graph;

(2) by the complement of the set of edges of obtained subgraph with respect to the regular graph with the given degree v and girth g.

The second stages is fulfilled by an algorithm of reduced search. Its implementation requires considerable computer time. One of the ways of reduction of graph's construction time is to use the properties of symmetry of these graphs. The aim of this paper is to present and investigate a subclass of $L(N,v,g)$ graphs - the $R_s(N,v,g)$ graphs that have a symmetry of connections and assume a parametric description. The parametrical description of the computer system structure is convenient for organization of operating system execution (under distribution and loading of the data, realization of routing procedures, etc.). It is more compact form of presentation of a network than description with the help of adjacency matrices or lists of structures.

2.1 Bounds on the diameter and the average distance of $L(N,v,g)$ graphs

Consider the properties of the graphs $L(v,g)$ and $L(N,v,g)$, which allow us, using the given values of N, v, and g, to determine the lower evaluations of their diameter and average distance. These values are true also for the graphs $R_s(N,v,g)$.

By definition, the diameter of the graph G $d = \max_{ij} d_{ij}$, where d_{ij} is the length of the shortest path from node i to node j. The average distance in the graph G $d_{av} = \sum_{ij} d_{ij}/N(N-1)$.

Lemma 1. *Let the graph $L(v,g)$ have the degree $v > 3$ and the girth $g > 3$ or $v = 3$ and $g \ge 6$. If the node $a \in A_k$, $k > 0$, then a is adjacent either to one node on the layer $k-1$ and to $v-1$ on the layer $k+1$, or to two nodes on the layer $k-1$ and to $v-2$ on the layer $k+1$, or if g is odd, to one node on the layer $k-1$, to one on the layer k, and to $v-2$ on the layer $k+1$.*

Let x_k, z_k, y_k denote the numbers of nodes situated on the layer k, $k > 0$, and having the above-mentioned types of connection, in the order they were listed in Lemma 1.

Corollary. *Let the graph $L(v,g)$ have the degree $v > 3$ and the girth $g > 3$. Then the number of nodes situated on the layer k, $k > 0$, is determined from the following formulas: $n_k = x_k + y_k + z_k$ for g odd, $n_k = x_k + z_k$ for g even.*

The proofs of Lemma 1 and Theorems 1 and 2 can be found in Korneyev and Monakhov (1980).

Now we find the distribution of x_k, y_k z_k depending on the values of k, v, and g.

Theorem 1. *Let the graph $L(v,g)$ have the degree $v > 3$ and the girth $g > 3$, or $v = 3$ and $g \ge 6$, and let $g = 2n+1$, where $n \ge 2$ is the natural number. Then the distribution of x_k, y_k, z_k depending on the number of layer k is determined by the following recurrent relationships:*

1. $x_0 = 0, y_0 = 0, z_0 = 0$ *for* $k = 0$;
2. $x_1 = v, y_1 = 0, z_1 = 0$ *for* $k = 1$
3. $x_k = x_{k-1}(v-1), y_k = 0,$ $z_k = 0$ *for* $1 < k \le n-1$;
4. $x_k = x_{k-1}(v-1) - 2v, y_k = 2v,$ $z_k = 0$ *for* $k = n$;
5. $x_k = x_{k-1}(v-1) + (y_{k-1} + z_{k-1})(v-2) - 2z_k - y_k,$ $y_k = 2x_{k-n}(v-2) + 2(v-3)(y_{k-n} + z_{k-n}),$ $z_k = y_{k-n}/2$ *for* $k \ge n+1$.

Theorem 2. *Let the graph $L(v,g)$ have the degree $v > 3$ and the girth $g > 3$ or $v = 3$ and $g \ge 6$, and let $g = 2n$, where $n \ge 2$ is the natural number. Then the distribution of x_k and z_k depending on the number of layer k is defined by the following recurrent relationships:*

$x_0 = 0, z_0 = 0 \quad for \quad k = 0;$

2. $x_1 = v, z_1 = 0 \quad for \quad k = 1;$

3. $x_k = x_{k-1}(v-1), \ z_k = 0$
 $for \quad 1 < k \leq n-1;$

4. $x_k = x_{k-1}(v-1) - 2v, \ z_k = v$
 $for \quad k = n;$

5. $x_k = x_{k-1}(v-1) + z_{k-1}(v-2) - 2z_k,$
 $z_k = x_{k-n}(v-2) + z_{k-n}(v-3)$
 $for \quad k \geq n+1.$

Theorems 1 and 2 allow us to determine the lower bounds on the diameter and the average distance of $L(N, v, g)$ graphs. Denote these evaluations by d^* and d^*_{av}, respectively. These values are found from the following relations:

$\sum_{k=0}^{d^*-1} n_k < N \leq \sum_{k=0}^{d^*} n_k, \ d^*_{av} = \sum_{k=1}^{d^*} kn_k/N,$ (1)

where $n_k = x_k + y_k + z_k$.

It follows from Theorems 1 and 2 that on all the layers of the optimal $L(N, v, g)$ graph up to the layer $[g/2]$ the number of nodes is equal to the maximal possible one for the given value of v. The optimal graph is chosen among $L(N, v, g)$ graphs with the maximal possible girth g for the given N and v. These structures have the minimal possible diameter and the minimal possible average distance among all the graphs with N nodes whose degree is equal to v.

3 Definition and properties of $R_s(N, v, g)$ graphs

We now distinguish a subclass of the graphs $L(N, v, g)$ that has a symmetry of connections, and consider its properties.

Let $R_\mu(N, v, g)$ be a subclass from the class of $L(N, v, g)$ graphs with the set of nodes $V = \{1, 2..., N\}$, the set of edges $E \subseteq V^2$, the group of automorphisms $Aut(R)$, and the equivalence relation μ forming a partition of the set of nodes V into $m \leq N$ classes V_i, such that for each pair of nodes $k, \quad j \in V_i, \quad i = \overline{1, m}$, an automorphism $\varphi \in Aut(R)$ exists, that transforms k to j:

$$\forall (k, j \in V_i) \exists (\varphi \in Aut(R))(\varphi(k) = j).$$

The equivalence μ on the set of nodes V of the graph $R_s(N, v, g)$, which will be considered further, is the congruence modulo of some natural number s dividing completely N, i.e.

$$\mu = \{(a, b) \in V^2 \quad a \equiv b (\mathrm{mod} \ s)\}, \qquad (2)$$

where $s \leq N, \quad N \equiv 0(\mathrm{mod} \ s)$.

In this case, if the equivalence μ is defined by expression (2), we denote $R_\mu(N, v, g)$ graphs in terms of $R_s(N, v, g)$ *graphs.*

Distinguish the class of $R_s(N, v)$ graphs that includes all $R_s(N, v, g)$ graphs with fixed values of s, N and v. Thus, the total set of nodes V of the graph $R_s(N, v)$ is subdivided into s equivalence classes V_i:

$$V_i = \{a \mid a \in V, \quad a \equiv i(\mathrm{mod} \ s)\},$$

where $i = \overline{1, s}$.

Let $r = N/s$. It follows from the definition of $R_\mu(N, v, g)$ graphs that for each pair of nodes $a, b \in V_i$, $i = \overline{1, s}$, of the graph $R_s(N, v)$ an automorphism $\varphi \in Aut(R)$ exists such that $\varphi(a) = b$, that is:

$$b \equiv a + js(\mathrm{mod} \ N), \quad j = \overline{1, r}. \qquad (3)$$

If in the graph $R_s(N, v)$ two nodes a and c are connected by the edge $(a, c) \in E$ and $c - a \equiv l(\mathrm{mod} \ N)$, where l is a natural number and $l < N$, then we call l a mark of the edge (a, c). Note, the edge has also the mark $l' \equiv a - c(\mathrm{mod} \ N)$.

Lemma 2. *If two nodes $a \in V_i, \quad i \in \overline{1, s}$, and $c \in V$ of the graph $R_s(N, v)$ are connected by the edge $(a, c) \in E$ with the mark l, then the edge $(b, d) \in E$ with the mark l is incident to each node $b \in V_i$.*

Proof. Since $a, \ b \in V_i$ it is seen from (3) that there exists an automorphism of the graph $R_s(N, v)$ such that $\varphi(a) = b$ and congruence (3) is true. Let $\varphi(c) = d$, then we have

$$d \equiv c + js(\mathrm{mod} \ N), \quad j = \overline{1, r} \qquad (4)$$

If $(a, c) \in E$, we obtain $(\varphi(a), \varphi(c)) \in E$, i.e. $(b, d) \in E$. From expressions (3) and (4) we have $d - b \equiv c - a(\mathrm{mod} \ N)$, and since $c - a \equiv l(\mathrm{mod} \ N)$, then $d - b \equiv l(\mathrm{mod} \ N)$. **Q.E.D.**

Corollary. *Let in the graph $R_s(N, v)$ the node $a \in V_i, \ i \in \overline{1, s}$. We denote by $L_i = \{l_{ik}\}, \quad i \in \overline{1, s}, \quad k \in \overline{1, v}$, the set of marks of edges incident to the node a. Then the set of marks of the edges incident to any node $b \in V_i$, is L_i.*

We call the set $L = \{l_{ik}\}, \quad i = \overline{1, s}, \quad k = \overline{1, v}$, a set of marks of edges of the graph $R_s(N, v)$. Two nodes a and b of the graph $R_s(N, v)$ are connected by the edge $(a, b) \in E$, if and only if there exists a natural number $l_{i,k} < N$, where $l_{ik} \in L, \quad i \in \overline{1, s}, \quad k \in \overline{1, v}$, such that if $a \equiv i(\mathrm{mod} \ s)$, then $b - a \equiv l_{ik}(\mathrm{mod} \ N)$, i.e.

$$(a, b) \in E \Leftrightarrow (\exists l_{ik} \in L)(a \equiv i(\mathrm{mod} \ s))\&$$

$$\&(b - a \equiv l_{ik} (\mathrm{mod}\ N)).$$

Thus, if the number of nodes N, the number of equivalence classes s, and the set of marks L are given, then $R_s(N, v, g)$ graph is completely defined.

Let E_{ik} denote the set of edges with the mark l_{ik}:

$$E_{ik} = \{(a, b) \in E \mid a \equiv i (\mathrm{mod}\ s),$$

$$b \equiv a + l_{ik} (\mathrm{mod}\ N)\},$$

where $i \in \overline{1,s}$, $k \in \overline{1,v}$. Edges from the set E_{ik} have also the mark

$$l_{jm} = N - l_{ik}, \qquad (5)$$

where $j \equiv i + l_{ik} (\mathrm{mod}\ s)$, $i, j \in \overline{1,s}$, $k, m \in \overline{1,v}$, and, hence, the sets E_{jm} and E_{ik} coincide.

Let L^* denote the minimal necessary set of marks. In order to go from the set L to the set L^*, we must to delete from the set L one mark from each pair of marks connected by relation (5). For the inverse transition from L^* to L it is necessary to find an additional mark by relation (5) for each mark from L^*.

Example 1. Let us consider Petersen's graph (Figure 2), in which $N = 10$, $v = 3$, $g = 5$, $s = 2$, i.e., it is the graph $R_2(10, 3, 5)$ with the set of marks $L = \{1, 2, 8;\ 4, 6, 9\}$ (the minimal set of marks $L^* = \{1, 2;\ 4\}$).

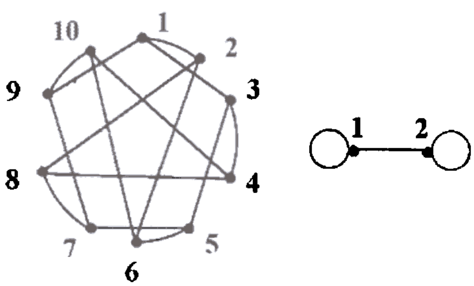

Figure 2: Petersen's graph and its $H(R)$ graph

Let us present some properties of the graphs $R_s(N, v)$:

1. $V = \bigcup_{i=1}^{s} V_i$, $V_i \bigcap V_j = \emptyset$ for $i \neq j$, $\mid V_i \mid = r$, where $i = \overline{1,s}$.

2. $E = \bigcup_{i=1}^{s} \bigcup_{k=1}^{v} E_{ik}$, $\mid E_{ik} \mid = r$, where $i = \overline{1,s}$, $k = \overline{1,v}$, $\mid E \mid = srv/2$.

3. If the nodes a, $b \in V$ of the graph $R_s(N, v)$ are connected, then

$$b - a \equiv \sum_{i=1}^{s} \sum_{k=1}^{v} l_{ik} t_{ik} (\mathrm{mod}\ N),$$

where t_{ik} is the number of edges with the mark l_{ik}, which belong to the path from a to b.

The graph $R_s(N, v, g)$ (as it has been shown in Monakhov (1979)), is described by a semigroup and it is isomorphic to the graph of the semigroup of transformations of the classes of equivalence.

3.1 Connectivity of $R_s(N, v)$ graphs

As far as we are interested only connected graphs as interconnection networks of multicomputer systems let us consider the question on the connectivity of the graphs $R_s(N, v)$. Let $r = N/s$. Let $H(R)$ denote a graph with the set of nodes $VH = \{1, 2 ..., r\}$ and the set of edges EH which is obtained from the graph $R_s(N, v)$ with the set of marks L for homomorphism $\phi : i \to j$, where $i \in V$, $j \in VH$, and $j = \left[\frac{i-1}{s}\right] + 1$; $[x]$ is integral part of x. Here

$$(a, c) \in EH \Leftrightarrow (\exists l_{mk} \in L)$$

$$(c \equiv a + b_{mk} (\mathrm{mod}\ r), \qquad (6)$$

$$b_{mk} = \left[\frac{d_{mk} - 1}{s}\right], \ d_{mk} \equiv m + l_{mk} (\mathrm{mod}\ N)),$$

where $m \in \overline{1,s}$, $k \in \overline{1,v}$, $d_{mk} \in \overline{1,N}$, $b_{mk} \in \overline{1,r}$.

Denote $B = \{b_{mk}\}$, $m = \overline{1,s}$, $k = \overline{1,v}$, and rewrite expression (6) in the form:

$$(a, c) \in EH \Leftrightarrow (\exists b_{mk} \in B)(c \equiv a + b_{mk} (\mathrm{mod}\ r)),$$

where $m \in \overline{1,s}$, $k \in \overline{1,v}$.

Note that the obtained definition of the graphs $H(R)$ coincides with that of circulant graphs with the set B as a set of marks provided all closed loops are eliminated.

Using property on connectivity of circulants (Vorobjev, 1974), we obtain the condition of connectivity of the graph $H(R)$.

Lemma 3. *To make the graph $H(R)$ with the number of nodes r and the set of marks B a connected graph, it is necessary and sufficient that the numbers $\{r, B\}$ be mutually prime.*

Example 2. For Petersen's graph (see Figure 2) $H(R)$ graph is complete graph K_5 with the number of nodes $r = 5$. In this case $B = \{1, 2, 3, 4\}$.

Let $\Gamma(R/s)$ denote a graph with the set of nodes $V\Gamma = \{1, 2 ..., s\}$ and the set of edges $E\Gamma$, obtained from the graph $R_s(N, v)$ with the set of marks L for homomorphism $\epsilon : i \to j$, where $i \in V$, $j \in V\Gamma$ and $i \equiv j (\mathrm{mod}\ s)$, here

$$(a, b) \in E\Gamma \Leftrightarrow (\exists l_{mk} \in L)(a + l_{mk} \equiv b (\mathrm{mod}\ s))\&$$

$$\&(a \equiv m(\mathrm{mod}\ s)),$$

where $m \in \overline{1,s}$, $k \in \overline{1,v}$. We assume that the set of marks of edges of the graph $\Gamma(R/s)$ coincides with the set of marks of the initial $R_s(N,v)$ graph.

Example 3. In Figure 2, right, the graph $\Gamma(R/2)$ of Petersen's graph is presented (see Example 1).

The graph $\Gamma(R/s)$ is a regular graph of the degree v if the initial graph was $R_s(N,v)$ with the set of marks L, except for the case when $l \in L$ and $l = N/2$. In this case the degree of a node incident to the edge with the mark l in the graph $\Gamma(R/s)$ is $v+1$.

Let us consider the connectivity of the graph $\Gamma(R/s)$ constructed from the graph $R_s(N,v)$ with the set of marks L. Let $L' \subset L$ be a subset such that for any $m \in \overline{1,s}$ and some $k \in \overline{1,v}$, if $l_{mk} \in L'$, then $l_{mi} \in L'$ for all $i \in \overline{1,v}$. Denote by I the set of first indexes of the marks $l_{mk} \in L'$, where $m \in \overline{1,s}$, $k = \overline{1,v}$.

Lemma 4. *The graph $\Gamma(R/s)$ is connected if and only if for each $L' \subset L$ there is a mark $l_{mk} \in L'$ such that $m + l_{mk} \equiv i(\mathrm{mod}\ s)$ and $i \notin I$ for $m,i \in \overline{1,s}$, $k \in \overline{1,v}$.*

Theorem 3. *The graph $R_s(N,v)$ is connected if and only if the graphs $H(R)$ and $\Gamma(R/s)$ are connected.*

3.2 Isomorphism of $R_s(N,v)$ graphs

Let us consider some sufficient conditions of isomorphism for the graphs $R_s(N,v)$.

Theorem 4. *The graphs $R_s(N,v)$ and $R'_s(N,v)$ with the sets of marks $L = \{l_{mk}\}$ and $L' = \{l'_{ik}\}$, respectively, are isomorphic if $l'_{ik} = N - l_{mk}$ for $i \equiv s + c - m(\mathrm{mod}\ s)$ for $i,m \equiv \overline{1,s}$, $k = \overline{1,v}$, $c \in \overline{0,s-1}$.*

Proof. The proof is based on the following: the mapping ψ is an isomorphism, where $\psi : a \to a'$, $a \in V$, $a' \in V'$, $a' \equiv N + d - a(\mathrm{mod}\ N)$, $d \equiv c(\mathrm{mod}\ s)$. **Q.E.D.**

Corollary. *The choice of the values of marks $l \in L^*$ of the graph $R_s(N,v)$ can be restricted by the interval from 1 to $s([r/2]+1)$, where $r = N/s$.*

Theorem 5. *The graphs $R_s(N,v)$ and $R'_s(N,v)$ with the sets of marks $L = \{l_{mk}\}$ and $L' = \{l'_{ik}\}$, respectively, are isomorphic if $l_{mk} = l'_{ik}$ for $m \equiv i \pm c(\mathrm{mod}\ s)$, where $i,m = \overline{1,s}$, $k = \overline{1,v}$, $c \in \overline{0,s-1}$.*

Proof. The proof is analogously to the proof of Theorem 4 under condition that we consider the isomorphism $\psi : a \to a'$, where $a \in V$, $a' \in V'$, $a' \equiv a + d(\mathrm{mod}\ N)$, *and* $d \equiv c(\mathrm{mod}\ s)$. **Q.E.D.**

Theorem 6. *The graphs $R_s(N,v)$ and $R'_s(N,v)$ with the sets of marks $L = \{l_{mk}\}$ and $L' = \{l'_{ik}\}$, respectively, are isomorphic if $l'_{ik} \equiv cl_{mk}(\mathrm{mod}\ N)$ provided that N and c are mutually prime numbers and $i \equiv cm(\mathrm{mod}\ s)$ for $i,m = \overline{1,s}$, $k = \overline{1,v}$.*

Proof. The proof is analogously to the proof of Theorem 4 under condition that we consider the isomorphism $\psi : a \to a'$, where $a \in V$, $a' \in V'$, $a' \equiv ca(\mathrm{mod}\ N)$, and c and N are mutually prime numbers. **Q.E.D.**

Along with the solution of the connectivity and isomorphism problems, the upper bound on girth of considered structures is defined in Monakhov (1979) by means of parametric description of such structures, at that the oppotunity of semigroup assignment of these structures is used.

3.3 Optimization problem for $R_s(N,v,g)$ graphs

The optimization problem considered in the paper is to found an optimal graph $R_s(N,v,g)$ having the minimal diameter and the minimal average distance for the given number of nodes N, degree v, girth g, and number of equivalence classes s. The given optimization problem is a problem of integer-valued programming with a nonlinear object function.

The following algorithms were proposed and realized for solving the problem: exhaustive search algorithm, an algorithm using the idea of branches and bounds, a genetic algorithm, simulating anealing and random search algorithm. These algorithms were used to obtain the graphs $R_s(N,v,g)$ for the values of $N \le 16384$, $v \le 12$, $g \le 8$, $s \le 4$. Parametric descriptions of some optimal or nearly optimal $R_s(N,v,g)$ graphs are given in Table 1.

In Figure 3 the graph $R_2(20,4,5)$ is presented. It has two equivalence classes and the set of marks $L = \{1,3,4,16;\ \ 8,12,17,19\}$.

4 Circulant graphs

Note, that for $g = 4$ the class of $R_s(N,v,g)$ graphs includes known network topologies such as hypercubes and circulant graphs.

The hypercubes can be described as $R_s(2^v,v,4)$ graphs with $s = 2^{v-2}$. For example, for $N = 2^3$ the

Table 1:

N	v	s	g	$d(D)$	$l_1, l_2, ..., l_{2v}$
64	6	2	5	3(3)	1 6 13 41 58 63 1 10 23 51 54 63
1024	10	2	4	4 (4)	45 246 277 460 479 543 564 593 701 778 68 308 323 431 481 545 716 747 956 979
2048	10	2	4	5 (4)	41 140 213 344 419 680 1368 1704 1863 1908 185 512 648 666 1382 1400 1536 1629 1835 2007
4096	12	2	4	5 (4)	48 194 1125 1231 1307 1483 1889 1959 1999 3671 3902 4048 342 425 1186 2097 2137 2207 2613 2789 2865 2910 2971 3754
8192	12	2	6	6 (4)	777 927 1508 2150 2157 2393 3971 4096 6042 6684 7019 7947 245 1173 1330 3990 4096 4202 4221 5799 6035 6862 7265 7415

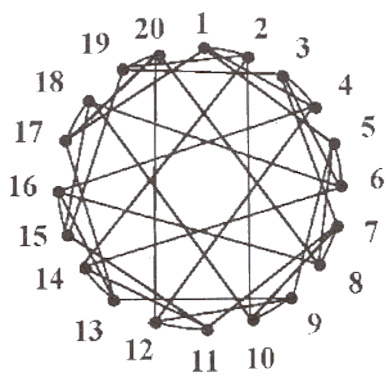

Figure 3: $R_2(20, 4, 5)$ graph

parametric description of the hypercube $R_2(2^3, 3, 4)$ has the form $\{1, 2, 6;\ 2, 6, 7\}$.

For $s = 1$ and $g = 4$ the class of $R_s(N, v, g)$ graphs includes the class of circulant graphs with the dimension equal to $v/2$, if the degree v is even.

Circulant graphs are intensively researched in computer science, graph theory and discrete mathematics. They are realized as interconnection networks in some computer systems (MPP, Intel Paragon, Cray T3D, etc.).

A *circulant* is an undirected graph $G(N; s_1, s_2, \dots, s_{v/2})$ with N nodes, labeled as $0, 1, 2, \dots, N-1$, having $i \pm s_1, i \pm s_2, \dots, i \pm s_{v/2}$ (mod N) nodes, adjacent to each node i.

The numbers $S = (s_i)$ $(0 < s_1 < \dots < s_{v/2} < N/2)$ are generators of the finite Abelian group of automorphisms connected to the graph. Circulant graphs $G(N; 1, s_2, \dots, s_{v/2})$, with the identity generator, are known as loop networks.

Let n_k determine the number of nodes on the layer k of the circulant G with the degree v, n_k^* being the upper bound for n_k. Let $u_k = \sum_{i=0}^{k} n_i$, denote the number of nodes in G, which are reachable by at most k steps from the node 0, and u_k^* being the upper bound for u_k. Recurrent relationships and formulas for calculation of n_k^* and u_k^* have been obtained in Wong and Coppersmith, 1974; Korneyev, 1974; Boesch and Wang, 1985:

$n_0^* = 1$,

$n_k^* = \sum_{i=0}^{v/2-1} \binom{v/2}{i} \binom{k-1}{v/2-i-1} 2^{v/2}$

$k \geq 1$;

$u_k^* = \sum_{i=0}^{v/2} \binom{v/2}{i} \binom{k}{v/2-i} 2^{v/2-i}$

A circulant graph G is called *extreme optimal* if $n_k = n_k^*$ for any $0 \leq k \leq d^* - 1$ and $n_{d^*} = N - u_{d^*-1}^*$, where the diameter d^* is defined by the correlation $u_{d^*-1}^* < N \leq u_{d^*}^*$. A graph G is called *optimal*, if $d(G) = d^*$. The diameter d^* is the exact lower bound for $d(N) = \min_S \{d(G(N; S))\}$.

Extreme optimal and optimal circulants have the minimum d (and the minimum d_{av} for extreme optimal circulants), the maximum reliability and connectivity (Korneyev, 1974; Boesch and Wang, 1985) and the minimum number of steps for a realization of communication algorithms (Monakhova and Monakhov, 1997) , but do not exist for some values N and $v > 4$ (Du et. al., 1990; Monakhova 1991; Bermond and Tzvieli, 1991).

Table 2

	Hypercube		Circulant			$L(N,v,g)$ graph			
$N=2^v$	$v=d$	d_{av}	v	$d^*(N)$	d^*_{av}	v	g	$d^*(N)$	d^*_{av}
64	6	3.0	6	4	2.5	6	6	3	2.29
256	8	4.0	8	4	3.3	8	6	3	2.7
512	9	4.5	8	5	4.02	9	6	3	2.81
1024	10	5.0	10	5	4.04	10	6	4	3.01
2048	11	5.5	10	6	4.7	11	6	4	3.47
4096	12	6.0	12	6	4.68	12	6	4	3.57
8192	13	6.5	12	6	5.34	13	6	4	3.78
16384	14	7.0	14	6	5.38	14	6	4	3.83
32768	15	7.5	14	7	6.09	15	6	4	3.89
65536	16	8.0	16	7	6.12	16	6	5	4.06

Table 3: Multidimensional circulants

N	v	$d(d^*)$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
1024	10	6 (5)	49	64	367	462	476			
2048	10	7 (6)	57	309	354	623	849			
4096	12	7 (6)	321	753	836	1380	1893	1990		
8192	12	8 (6)	151	860	1883	2203	2320	2454		
16384	14	8 (6)	490	1277	1645	2512	3832	4317	5448	
32768	14	9 (7)	195	1052	5398	5567	12171	12321	12989	

For $n=2$, an analytical solution of a problem of existence and synthesis of optimal circulants has been found in Monakhova, 1981; Boesch and Wang, 1985; Bermond et. al., 1985.

Theorem 7. *For any $N>4$ an extreme optimal circulant $G(N;s_1,s_2)$ exists and has the generators $\{s,s+1\}$, $s=\lfloor(\sqrt{2N-1}-1)/2\rfloor$, where $\lfloor x\rfloor$ is the nearest integer to x.*

In Du et. al., 1990; Bermond and Tzvieli, 1991; Monakhova, 1991; Mukhopadhyaya and Sinha, 1995 some conditions have been found for an existence of optimal loop circulant graphs with $v=4$, and dense infinite families of values of N, which are optimal, have been defined by analytical formulas. For $v>4$ these problems are known as NP - hard. Search algorithms and heuristics are used to synthesize nearly optimal circulants. The detailed review (without the results of Russian researches) of problems of a construction of optimal circulants (directed and undirected) and their generalizations is made in Bermond et. al., 1995.

5 Comparison of $R_s(N,v,g)$ graph subclasses by the diameter and the average distance

In this section the PRS graphs with the girth $g=6$ and multidimensional circulants are compared by the structural properties with a popular structure for parallel computer architectures such as hypercubes.

In Table 2 partial results of a comparison of hypercubes, $L(\text{or}R_s)(N,v,g)$ graphs ($g=6$) and circulants by the degree v, the diameter d and the average distance d_{av} for the same number of graph nodes $N=2^v$ are presented. The value of $d^*(N)$ is the exact lower bound on a diameter of $L(\text{or}R_s)(N,v,6)$ graphs. It is calculated from expression (1). In the case of circulants, the value of $d^*(N)$ is calculated from the expression for the values of u^*_k. For realization of a correct comparison a degree of a circulant is equal to a degree of a hypercube (or less by one unit).

In Figure 4 the diagrams are represented for the diameter of hypercubes and exact lower bounds on diameters of $L(v,6)$ graphs and circulants as the functions of the values of N.

For all the values of $N\leq 2^{163}$ the character of variation of exact lower bound on a diameter of circulant graphs is expressed by the following approximating formula: $d^*(N)=a+b\ln N=a+b_1v$, where $a=2.123$, $b=0.467$, $b_1=0.324$.

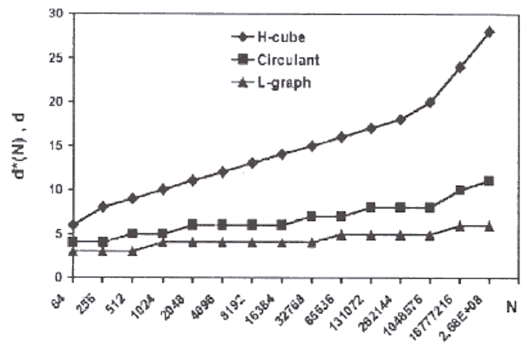

Figure 4: The diameters of graphs

For hypercubes the diameter $d = a_2 + b_2 \ln N$, where $a_2 = 3.639$, $b_2 = 1.443$.

Parametrical descriptions of the really existing nearly optimal circulant graphs for some values of N and v from Table 2, obtained by means of genetic algorithm (Monakhova et. al., 1999), are represented in Table 3. We see that the diameters of graphs for adduced values of N exceed the exact lower bounds by two units at most.

Thus circulant graphs have the same degree (or less by one unit), and a smaller logarithmic diameter (by a factor of 1/3) and smaller average interprocessor distance than hypercubes for the same number of processors under implementation as interconnection networks in parallel computer systems. Appropriate parameters for $R_s(N, v, g)$ graphs with $s > 1$ and $g > 4$ are better: obtained evaluations show the diameter of optimal PRS graphs is $\approx 0.21 \log_2 N$ as compared with $\log_2 N$ of the diameter of hypercube.

6 Conclusion

We have presented a new class of undirected parametrically prescribed regular networks with logarithmic estimate on the diameter for multicomputer parallel architectures. The class of PRS networks possesses broad possibilities for parameter variations and obtaining a spectrum of networks with different properties. The considered generalized class of structures are proved to be better than hypercube topology with respect to the diameter and the average distance. Therefore they require a considerably smaller number of interprocessor changes for the realization of collective and individual change schemes such as gossiping, broadcasting and routing.

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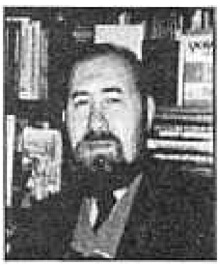

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